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ELEMENTS OF THE THEORY OF INFINITE PROCESSES

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ELEMENTS OF THE THEORY OF INFINITE PROCESSES

BY

LLOYD L. SMAIL, PH.D.
Professor of Mathematics, Lehigh University

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PREFACE

Under the title of Infinite Processes, I have included the subjects of Infinite Sequences, Infinite Series, Infinite Products, Infinite Continued Fractions, Infinite Determinants, and Infinite Integrals. The theory of convergence of infinite forms is of the greatest importance in the study of modern mathematical analysis. Notwithstanding its importance, no suitable introductory text-book on the subject is available. It was to fill this need that this book was written. The book is planned for an introductory course, giving the more important conceptions and propositions of the field of infinite processes, in sufficient detail to be intelligible to a beginner in the subject, and serving as an introduction to a more detailed and complete study of the special fields. No attempt has been made to go deeply into any particular aspect of the various subjects; the aim has been merely to present the fundamental ideas at the basis of the special subjects. The references at the end of each chapter will guide the student into further study of any particular topic.

The only prerequisite for the study of this book is a good course in the Calculus. It is suitable, therefore, for use either in advanced undergraduate courses, or in graduate courses. The material included covers not only the usual discussion of infinite series and products and their convergence and manipulations, but also the more important conceptions and propositions in the less familiar fields of infinite continued fractions, infinite determinants, infinite integrals, Dirichlet series, Fourier series, asymptotic and summable series, and in a final chapter, the definitions and a few fundamental properties of the more important higher transcendental functions defined by infinite processes.

I am indebted for material to many sources, but particularly to Pierpont's "Theory of Functions of Real Variables," Bromwich's "Theory of Infinite Series," and Whittaker and Watson's "Modern Analysis."

LLOYD L. SMAIL.

UNIVERSITY OF WASHINGTON,
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ELEMENTS OF THE THEORY OF INFINITE PROCESSES

CHAPTER I

INFINITE AGGREGATES

1. Infinite Aggregate.—An aggregate of numbers is a collection or set of real or complex numbers. An aggregate is defined when a law or rule is given by which it is determined whether any given number belongs to the aggregate. An infinite aggregate is one which contains an unlimited number of elements.

By virtue of the one-to-one correspondence between the real numbers and the points of a straight line,[1] we may represent real numbers by points of a straight line, and complex numbers by points of a plane, and we may use the terms "number" and "point" interchangeably.

2. Bounded Aggregate.—*An aggregate of real numbers is said to be* **bounded above** *when a number N can be found such that for every element x of the aggregate, we have $x < N$. We say that the aggregate is* **bounded below** *when a number M can be found such that $x > M$ for every element x of the aggregate.* An aggregate is called simply **bounded** when it is bounded above and below. An aggregate is unbounded when it is not possible to find such numbers M and N. When the aggregate is unbounded above, for every positive number G, as large as we please, there are elements of the aggregate which are $> G$; and when the aggregate is unbounded below, there are elements of the aggregate which are $< -G$.

---

[1] See Pierpont, "Theory of Functions of Real Variables," vol. I, pp. 78–79.

An aggregate of complex numbers is said to be **bounded** when it is possible to enclose all the points of the aggregate within a circle of finite radius with center at the origin; or in other words, when it is possible to find a positive number ρ such that the absolute values of all the numbers of the aggregate are less than ρ: for every complex element z of the aggregate, we have $|z| < \rho$. If there exist points of the aggregate outside every circle of radius however great about the origin, then the aggregate is unbounded. Evidently, if the elements z of the aggregate are of the form $z = x + iy$, then the aggregates of real elements x and y are bounded when the complex aggregate is bounded.

Examples.—The set of all positive proper fractions is a bounded aggregate; the set of all positive integers is an aggregate bounded below but unbounded above; and the set of all irrational numbers is an unbounded aggregate.

3. Theorem.—*Let E_1 and E_2 be two aggregates of real numbers, satisfying the following conditions:*

1° *Every number of E_1 is $<$ every number of E_2;*

2° *For any real number $\epsilon > 0$, however small, there exist elements e_1 in E_1 and e_2 in E_2 such that $e_2 - e_1 < \epsilon$. Then these two aggregates determine a real number α.*

For, let us put all rational numbers which are $\leqq$ a number of E_1 in a class A_1, and all rational numbers which are $\geqq$ a number of E_2 in a class A_2.

First, suppose there is *no* rational number which is not in either A_1 or A_2. Then *every* rational number is classified, and we have a separation or Dedekind cut (A_1, A_2) defining a real number,[1] which is the number α of the theorem.

Next, suppose that there is *one* and only one rational number c which is not in either A_1 or A_2. If we put this rational number c in either A_1 or A_2, we again have a separation of all the rational numbers; this cut defines the rational number c, which is the number α of the theorem.

---

[1] According to the Dedekind theory of irrational numbers, if we separate all the rational numbers into two classes A_1, A_2, such that: (1) every rational number belongs to A_1 or A_2, and (2) every number of A_1 is $<$ every number of A_2, then this separation or so-called "cut" (A_1, A_2) defines a real number, rational or irrational depending on

Finally, suppose there are *two* distinct rational numbers c_1 and c_2 which do not belong to either A_1 or A_2. Then c_1 and c_2 are $>$ every element e_1 of E_1 and are $<$ every element e_2 of E_2. Let e'_1 and e'_2 be chosen so that $e'_2 - e'_1 < \epsilon$. Then $c_1 > e'_1$, $c_2 > e'_1$, $c_1 < e'_2$, $c_2 < e'_2$, so that $|c_1 - c_2| < e'_2 - e'_1 < \epsilon$. But since ϵ can be chosen arbitrarily small, this last inequality is impossible if c_1 and c_2 are distinct numbers. It follows that two such numbers c_1 and c_2 cannot exist.

Hence, we have shown that in every case the two aggregates E_1 and E_2 determine a unique real number α.

4. Limiting Point.—By a limiting point of an aggregate, we mean a point about which an infinite number of points of the aggregate cluster. In more precise arithmetic terms, *a point z_0 is said to be a* **limiting point** *of an aggregate E if for every $\epsilon > 0$, however small, there are an infinite number of elements z of E such that $|z - z_0| < \epsilon$.*

Geometrically, for x_0 to be a limiting point of a real aggregate, there must be an infinite number of points x of the aggregate in every interval about the point x_0, however small; and for z_0 to be a limiting point of a complex aggregate, there must be an infinite number of points z in every circle, however small, about z_0 as center.

Examples.—The aggregate of numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ has the limiting point 0. The aggregate of numbers $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}, \ldots$ has the limiting points $+1$, -1.

---

whether class A_1 has a greatest or class A_2 a least number, or neither have a greatest or least number.

For a detailed discussion of irrational numbers, see HOBSON, "Theory of Functions of a Real Variable," chap. I, or CARSLAW, "Fourier's Series and Fourier's Integrals," chap. I, or PIERPONT, "Theory of Functions of Real Variables," vol. I, chap. II, or FINE, "College Algebra."

5. The Bolzano-Weierstrass Theorem.—*Every bounded infinite aggregate has at least one limiting point.*

Consider first the case of a real aggregate E.

Since the aggregate E is bounded, numbers M and N can be found such that $M < x < N$ for every x in E. Divide the interval (M, N) into two equal parts by the middle point $\frac{1}{2}(M + N)$; in at least one of these parts there must be an infinite number of elements of E; if there is only one such interval, denote it by (α_1, β_1), and if there are two such parts, denote either one of them by (α_1, β_1). Then bisect the interval (α_1, β_1), and in at least one of the halves there must be an infinite number of elements of E; denote this interval by (α_2, β_2). Bisect (α_2, β_2), and again there must be an infinite number of elements of E in one of these halves, call it (α_3, β_3). Continue this bisection indefinitely, and we obtain an infinite set of intervals

$$(\alpha_1, \beta_1),\ (\alpha_2, \beta_2),\ (\alpha_3, \beta_3),\ \ldots\ ,$$

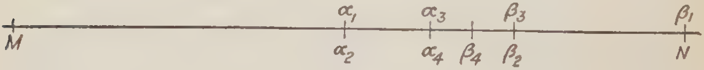

FIG. 1.

each of which is half the preceding and contains all the following intervals. This set of intervals gives us two infinite sets of points, the end-points of the intervals: the aggregate $\alpha_1, \alpha_2, \alpha_3, \ldots$ (call it E_1), and the aggregate $\beta_1, \beta_2, \beta_3, \ldots$ (call it E_2). Now these aggregates evidently satisfy the conditions of Art. **3**, for

$$\beta_n - \alpha_n = \frac{N - M}{2^n}.$$

Hence, the two aggregates E_1 and E_2 determine a point x_0, say, and about this point an infinite number of elements of E cluster, for this point lies in all the intervals (α_1, β_1), (α_2, β_2), (α_3, β_3), . . . Hence, this point x_0 is a limiting point of E. Thus, every bounded real aggregate has at least one limiting point. There is nothing in the above demonstration to prevent the aggregate from having more than this one limiting point.

Turning now to the case of a complex aggregate, the theorem may be proved by a method precisely similar to that for real aggregates just given, using rectangles instead of intervals. The given aggregate, being bounded, can be enclosed in a rectangle with sides parallel to the real and imaginary axes. Divide this rectangle into four equal rectangles by lines parallel to the sides, and continue this subdivision into fourths, just as before the successive intervals were bisected. In at least one of each of these successive sets of rectangles there are an infinite number of points of the aggregate, and these rectangles containing an infinite number of such points determine a point which will be a limiting point of the given aggregate.

6. Theorem.—*If the real aggregate* E *is bounded above, there exists a number* $\bar{B}$ *such that*

1° *No number of* E *is* $> \bar{B}$*;*

any number [illegible]

2° *Whatever be the number* $m < \bar{B}$, *however small the difference between* m *and* $\bar{B}$, *there is at least one element of* E *which is* $> m$.

If the real aggregate E *is bounded below, there exists a number* $\underline{B}$ *such that*

3° *No number of* E *is* $< \underline{B}$*;*

4° *Whatever be the number* $n > B$, *however small the difference between* n *and* $\underline{B}$, *there is at least one element of* E *which is* $< n$.

Consider first the case where E is bounded above.

If the aggregate has a greatest element, this is evidently the number $\bar{B}$ of our theorem.

If E has no greatest element, we determine the number $\bar{B}$ as follows. Make a cut in the rational number system by putting in the lower class all rational numbers which are $\leq$ a number of E, and in the upper class all rational numbers which are $>$ the numbers of E. There must exist numbers in the upper class since E is bounded above. We proceed to show that this cut defines a number which satisfies the conditions of the theorem.

First, the lower class has no greatest number. For let a be any number of the lower class; then either a is $<$ a number e of E or it is equal to a number e' of E. Suppose first that $a < e$; then the rational numbers between a and e belong to the lower class, and hence a cannot be the greatest number of the lower class. Suppose next that $a = e'$; since E has no greatest element, we can find in E a number $e'' > e'$, and then the rational numbers between e' $(=a)$ and e'' must belong to the lower class, and again a cannot be the greatest element of the lower class. Hence, the lower class has no greatest number.

If the upper class has a least number, this rational number is the number defined by the cut. If, however, the upper class has no least number, then this cut defines an irrational number. In either case, then, the cut defines a real number. Calling this number $\overline{B}$, we shall now show that it satisfies the conditions 1° and 2° of our theorem.

Since $\overline{B}$ is $>$ any number of the lower class, and if e is any element of E, we can find a number of the lower class which is $> e$, it follows that there is no number of E which is $> \overline{B}$, so that condition 1° is satisfied.

Now let m be any number $< \overline{B}$, but as near $\overline{B}$ as we please. Since the lower class contains the rational numbers $< \overline{B}$, there must be a number a of the lower class which is $> m$. Therefore, by our former argument, there must be a number e of E which is $> a$ and therefore is $> m$. Hence, condition 2° is satisfied.

The case of E bounded below can be treated in a precisely similar manner.

7. Upper and Lower Bounds of a Real Aggregate.—These numbers $\overline{B}$ and $\underline{B}$ are called the **upper bound** and **lower bound,** respectively, of the aggregate E, and may be denoted by $\overline{B}(E)$ and $\underline{B}(E)$, when we wish to indicate the aggregate to which they belong. Their definition may be stated as follows:

The **upper bound** *of a real aggregate E which is bounded above is the number $\overline{B}$* (whose existence has just been proved), *which satisfies the following conditions:*

1° *No element of E is $> \overline{B}$;*

2° *If ϵ be any positive number, as small as we please, there is at least one element of E which is $> \overline{B} - \epsilon$.*

The **lower bound** *of a real aggregate E which is bounded below is the number $\underline{B}$* (whose existence has just been proved), *which satisfies the following conditions:*

3° *No element of E is $< \underline{B}$;*

4° *If ϵ be any positive number, as small as we please, there is at least one element of E which is $< \underline{B} + \epsilon$.*

The upper and lower bounds are sometimes referred to together as the **extreme bounds** of the aggregate.

If the aggregate has a greatest number, this is evidently the upper bound, and it is then called the **maximum** of the aggregate. If the aggregate has a least number, this is the lower bound, and it is called the **minimum** of the aggregate. We may denote them by Max(E), and Min(E), respectively.

Examples.—(1) The aggregate of numbers $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots$ has the upper bound $\overline{B} = 1$ and lower bound $\underline{B} = 0$ (the minimum). This example shows that the upper bound need not belong to the aggregate.

(2) The aggregate of numbers $-1, +\frac{1}{2}, -\frac{1}{3}, +\frac{1}{4}, \ldots, (-1)^n \frac{1}{n}, \ldots$ has the lower bound $\underline{B} = -1$ and upper bound $\overline{B} = \frac{1}{2}$, both of which are attained; here -1 is the minimum and $\frac{1}{2}$ is the maximum.

8. Limiting Points and Bounds of a Real Aggregate.—By the Bolzano-Weierstrass theorem of Art. **5**, every bounded infinite aggregate has at least one limiting point. An aggregate may, however, have more than one limiting point; it may in fact have an infinite number.

Examples.—The aggregate of numbers of the type $\frac{1}{n}$ $(n = 1, 2, 3, \ldots)$ has one limiting point 0. The aggregate ½, ⅓, ⅔, ¼, ¾, ⅕, ⅘, . . . has the two limiting points 0 and 1. The aggregate of numbers of the type $\frac{1}{m} + \frac{1}{n}$ $(m, n = 1, 2, 3, \ldots)$ has an infinite number of limiting points, of the type $\frac{1}{n}$ $(n = 1, 2, 3, \ldots)$.

Evidently, if $\overline{B}$ and $\underline{B}$ are the upper and lower bounds of the aggregate, the limiting points must fall in the **closed** interval $(\underline{B}, \overline{B})$; by **closed** interval $(\underline{B}, \overline{B})$, we mean the set of points x such that $\underline{B} \leqq x \leqq \overline{B}$, while by an **open** interval $(\underline{B}, \overline{B})$, we mean the set of points x such that $\underline{B} < x < \overline{B}$.

The limiting points of an aggregate may or may not belong to the aggregate. In the examples above, the limiting points do not belong to the aggregate; in the aggregate 1, ½, ⅔, ¾, ⅘, ⅚, . . . , the limiting point 1 does belong to the aggregate. The system of real numbers is an aggregate in which each element is a limiting point.

9. Theorem.—*If the upper bound (or lower bound) of an aggregate does not itself belong to the aggregate, then it is a limiting point of the aggregate.*

For, if the upper bound $\overline{B}$ is not an element of the aggregate E, then by definition of upper bound, there are an infinite number of elements of E between $\overline{B}$ and $\overline{B} - \epsilon$, however small ϵ, for if there were only a finite number of them, then the greatest of them would be the upper bound of E; and hence $\overline{B}$ will be a limiting point of E. Similarly for the lower bound.

Example.—The aggregate 0, ½, ⅔, ¾, ⅘, . . . has the upper bound 1, which does not belong to the aggregate, but which is a limiting point.

The upper bound (or lower bound) *may be* a limiting point of the aggregate even when it belongs to the aggregate, as, for example, in the aggregate 0, ½, ⅓, ¼, . . .

10. Derived Aggregate of a Given Aggregate.—*The aggregate E' made up of all the limiting points of a given aggregate E is called the* **derived aggregate of E.** It may contain one point, a finite number of points, or an infinite number of points.

Examples.—For the aggregate $1, \frac{1}{2}, \frac{1}{3}\ \frac{1}{4}, \ldots, E'$ consists of the one point 0; for the aggregate $0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{4}{5}, \ldots, E'$ consists of the two points 0 and 1; while the aggregate $\frac{1}{m} + \frac{1}{n}$ $(m, n = 1, 2, 3, \ldots)$ has for its derived aggregate the set of numbers $\frac{1}{n}$ $(n = 1, 2, 3, \ldots)$, which forms an infinite aggregate.

11. Theorem.—*If E' is the derived aggregate of a bounded aggregate E, then every limiting point of E' belongs to E'.*

For, let ϵ be any arbitrarily small positive number; then if a is a limiting point of E', by definition, we can find an element b of E' such that $|a - b| < \frac{\epsilon}{2}$. But since b is an element of E' different from a, it is a limiting point of E different from a, and hence there must be an infinite number of elements e of E such that $|b - e| < \frac{\epsilon}{2}$. From these two inequalities we get by addition, $|a - e| < \epsilon$ for an infinite number of elements e, which shows that a is a limiting point of E and therefore belongs to E'.

12. Theorem.—*If E' is the derived aggregate of a bounded real aggregate E, then E' has an upper bound, $\overline{B}'$ say, which belongs to E'; that is, the upper bound $\overline{B}'$ of E' is the maximum of E'; and also the lower bound $\underline{B}'$ of E' is the minimum of E'.*

For, if the upper bound $\overline{B}'$ of E' did not belong to E', it would be a limiting point of E' (Art. **9**), and then by Art. **11** it would belong to E', contrary to hypothesis. That the aggregate E' has an upper bound $\overline{B}'$ is evident since a limiting point of E cannot exceed the upper bound of E, and hence E' is bounded.

Similarly for the lower bound.

13. Upper and Lower Limits of a Bounded Real Aggregate.—*The upper bound* $\overline{B}'$ *of the derived aggregate* E' *of a given bounded real aggregate* E, *that is, the maximum limiting point of* E, *is called the* **upper limit** *of* E; *and the lower bound* $\underline{B}'$ *of the derived aggregate* E', *or the minimum limiting point of* E, *is called the* **lower limit** *of* E.

The upper and lower limits of an aggregate are together called the **extreme limits.**

The upper limit of E may be denoted by $\overline{L}(E)$, and the lower limit by $\underline{L}(E)$.

Every bounded real aggregate has an upper limit and a lower limit.

Examples.—(1) For the aggregate $1 + (-1)^n \cdot \dfrac{n-1}{n}$ $(n = 1, 2, 3, \ . \ . \ . \ )$, we have $\overline{B} = 2$, $\underline{B} = 0$, the limiting points are 0, 2, and hence $\overline{L} = 2$, $\underline{L} = 0$. In this case, the upper bound and limit, and lower bound and limit coincide respectively.

(2) For the aggregate $1 + \dfrac{1}{n} + (-1)^n \cdot \dfrac{n-1}{n}$ $(n = 1, 2, 3, \ . \ . \ . \ )$, we have $\overline{B} = 2$, $B = 0$, the limiting points are 2 and 0, and hence $L = 2$, $\underline{L} = 0$.

(3) For the aggregate $(-1)^{n-1}\left(1 + \dfrac{1}{n}\right)$, we have $\overline{B} = 2$, $\underline{B} = -3/2$, $\overline{L} = 1$, $\underline{L} = -1$.

14. Theorem.—*The upper and lower limits* $\overline{L}$ *and* $\underline{L}$ *of a bounded real aggregate* E *possess the following properties:*

If ϵ, ϵ' *are any arbitrarily small positive numbers, then*

1° *There are only a finite number of elements* x *of* E *such that* $x > \overline{L} + \epsilon$, $x < \underline{L} - \epsilon$;

2° *There are an infinite number of elements* x *of* E *such that* $x > \overline{L} - \epsilon'$, $x < \underline{L} + \epsilon'$.

Consider the upper limit $\bar{L}$. There can only be a finite number of elements x of E which are $> \bar{L} + \epsilon$, for otherwise there would be a limiting point of E in the interval $(\bar{L} + \epsilon, \bar{B})$, and hence a number of E' which is $> \bar{L}$, which is impossible since $\bar{L}$ is the maximum element of E'. On the other hand, there are an infinite number of elements of E which are $> \bar{L} - \epsilon'$, since $\bar{L}$ is a limiting point of E, and hence there must be an infinite number of elements of E in the interval $(\bar{L} - \epsilon', \bar{L} + \epsilon')$.

Similarly we prove the theorem for the lower limit $\underline{L}$.

15. Theorem.—*If a bounded real aggregate E has no maximum or minimum, then its upper bound and upper limit are identical, and its lower bound and lower limit are identical.*

For, if E has no maximum, its upper bound $\bar{B}$ must be a limiting point of E (Art. **9**), and hence $\bar{B}$ must be the greatest limiting point of E, and therefore coincides with $\bar{L}$, the upper limit, which is also the greatest limiting point. Similarly for the lower bound and lower limit.

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CHAPTER II

INFINITE SEQUENCES

16. Infinite Sequence.—*If an infinite aggregate of real or complex elements is such that we can establish a* 1-1 *correspondence between the elements of the aggregate and the set of positive integers* 1, 2, 3, . . . , *n*, . . . , *we call the aggregate an* **infinite sequence.** An infinite sequence may then be represented by

$$u_1, u_2, u_3, \quad . \quad . \quad . \quad , u_n, \quad . \quad . \quad . \quad ,$$

where the u's are real or complex numbers; or, more briefly, by (u_n).

An infinite sequence is considered as **defined** when some law or rule is given by which we may determine the n^{th} term when the index n is assigned.

A sequence is usually given by a formula for its n^{th} term, or by means of the first few terms, from which the n^{th} term may be found, or by a combination of both methods.

Examples.—(1) In the sequence $1/2$, $-2/3$, $3/4$, $-4/5$, . . . , the n^{th} term is $u_n = (-1)^{n+1} \cdot \dfrac{n}{n+1}$.

(2) If $u_n = \dfrac{n^2}{n+1}$, the first few terms are $1/2$, $4/3$, $9/4$, $16/5$, . . .

17. Bounded Sequence.—An infinite sequence is said to be **bounded** when the aggregate made up of the distinct terms of the sequence is bounded. Every bounded real sequence must have an upper bound and a lower bound, and *if* $\overline{B}$ *and* $\underline{B}$ *are these bounds, they possess the following properties:*

1° *For no index n is* $u_n > \overline{B}$ *nor* $u_n < \underline{B}$;

2° *For at least one index n,* $u_n > \overline{B} - \epsilon$ *and* $u_n < \underline{B} + \epsilon$, *however small* $\epsilon > 0$ *may be.*

The aggregate of distinct terms of a bounded sequence must have at least one limiting point (Art. **5**), and this limiting point is called a limiting point of the sequence.

18. Extreme Limits of a Real Sequence.—The **upper** and **lower limits** of a sequence are the upper and lower limits of the corresponding aggregate; or, as we may say, they are the maximum and minimum limiting points of the sequence. Every bounded real sequence must have an upper and lower limit. They may be denoted by $\overline{L}(u_n)$ and $\underline{L}(u_n)$ respectively; if we wish to indicate that n is the index of the sequence, we write $\overline{L}_{n} u_n$ and $\underline{L}_{n} u_n$ respectively. The symbol $\overline{\underline{L}} u_n$ may be used when either of the extreme limits may be taken.

The extreme limits of a bounded real sequence possess the following properties:

For the upper limit $\overline{L}$:

1° *An index* n_0 *can be found such that for every* $n > n_0$,

$$u_n < \overline{L} + \epsilon;$$

2° *For infinitely many indices n, we have*

$$u_n > \overline{L} - \epsilon.$$

For the lower limit $\underline{L}$:

3° *An index* n'_0 *exists such that for every* $n > n'_0$,

$$u_n > \underline{L} - \epsilon;$$

4° *For infinitely many indices n, we have*

$$u_n < \underline{L} + \epsilon.$$

These results follow directly from Art. **14**.

Some writers use the notation "$\overline{\lim}\, u_n$" or "lim sup u_n" for upper limit, and "$\underline{\lim}\, u_n$" or "lim inf u_n" for lower limit.

19. Limit of a Sequence.—*If a sequence of real or complex terms is bounded, and has* **one** *and* **only one** *limiting point, this limiting point is called* **the limit** *of the sequence.*

Suppose the sequence is (u_n), and let U be its one limiting point. Then by definition of limiting point, for any $\epsilon > 0$, however small, we can find an infinite number of elements u_n of the sequence such that $|U - u_n| < \epsilon$, and hence we can find an index ν (finite) such that for every $n > \nu$, this inequality is satisfied. The definition of limit of a sequence may then be stated:

The sequence (u_n) *has a* **limit U** *when, for every* $\epsilon > 0$, *however small, a finite index* ν *exists such that for every* $n > \nu$,

$$|U - u_n| < \epsilon.$$

We write $U = \underset{n}{L}\, u_n$, or when no confusion can arise as to which variable or index increases indefinitely, $U = Lu_n$ or $u_n \rightarrow U$.

Many writers use the notation $U = \lim\limits_{n \to \infty} u_n$ or $U = \lim\limits_{n = \infty} u_n$.

The number U is finite for any bounded sequence.

The following abbreviated notation[1] for the definition of limit will be found very convenient:

$$\epsilon > 0, \quad \overset{\text{index}}{\nu}, \quad |\overset{\text{limit}}{U} - u_n| < \epsilon, \quad n > \nu,$$

is to mean: for each positive ϵ (as small as we please), there exists an index ν such that $|U - u_n| < \epsilon$ for every $n > \nu$.

Examples.—(1) The sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ . \ . \ . \ , \frac{1}{n}, \ . \ . \ .$ has the limit $U = 0$.

(2) The sequence $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ . \ . \ . \ , \frac{n-1}{n}, \ . \ . \ .$ has the limit $U = 1$.

(3) The sequence $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ . \ . \ . \ , (-1)^{n+1}\frac{1}{n}, \ . \ . \ .$ has $Lu_n = 0$.

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[1] Due to PIERPONT, *cf.* his "Theory of Functions of Real Variables," vol. I, art. **42**.

If $u_n \rightarrow U$ and we put $|U - u_n| < \epsilon$ for $n > \nu$, the index ν varies with ϵ, and increases indefinitely as $\epsilon \rightarrow 0$, but it must be determinate for each positive value of ϵ.

20. Unbounded Sequences. Infinite Limits.—Let (u_n) be an unbounded real sequence. The following cases may occur:

(1) After a certain term, the terms u_n are all positive, and if G is an arbitrarily large positive number, there exists an index ν such that for every $n > \nu$, we have $u_n > G$. In this case we say that the sequence has the **limit plus infinity,** and we write

$$\underset{n}{L} u_n = +\infty, \text{ or } Lu_n = +\infty, \text{ or } u_n \rightarrow +\infty.$$

(2) After a certain term, the terms u_n all become negative, and for every $G > 0$, however large, an index ν exists such that for every $n > \nu$, we have $u_n < -G$. We say in this case that the sequence has the **limit minus infinity,** and we write

$$\underset{n}{L}\, u_n = -\infty, \text{ or } L\, u_n = -\infty, \text{ or } u_n \rightarrow -\infty.$$

In both these cases we say that the limit is **properly infinite.**

(3) The terms do not finally have the same sign, but still $\underset{n}{L}\, |u_n| = +\infty$, that is, for every $G > 0$, however large, an index ν exists such that for every $n > \nu$, we have $|u_n| > G$. In this case, we say that the sequence has a **limit improperly infinite.**

In all three of these cases, we call the limits **infinite limits.**

Examples.—The sequences 1, 3, 5, 7, . . . , and 1!, 2!, 3!, . . . have the limit $+\infty$; the sequence -1, -4, -9, -16, . . . has the limit $-\infty$; and the sequence 1, -2, 3, -4, . . . has a limit improperly infinite.

The preceding discussion applies only to real sequences. If (u_n) is an unbounded sequence of complex numbers, we have for every $n > \nu$, $|u_n| > G$, however large $G > 0$ may be. In this case, we simply say that the sequence has an **infinite limit,** or that it has the **limit infinity,** and write $L\, u_n = \infty$.

It should be noted that the symbols $+\infty$ and $-\infty$ are not numbers, but are merely convenient symbols to express certain modes of variation.

21. Graphical Representation.—Suppose that (u_n) is a real sequence which has the limit U.

On a horizontal axis with origin at 0, lay off the points u_n, and also the point U and on each side of U the points $U - \epsilon$ and $U + \epsilon$ at a distance ϵ away; these last two points determine an interval of length 2ϵ, marked heavy in the figure, which may be called the ϵ-interval. Then for each small ϵ-interval, there must exist an index ν such that the points $u_{\nu+1}, u_{\nu+2}, \ldots, u_n, \ldots (n > \nu)$ all fall within this interval.

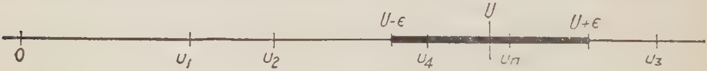

FIG. 2.

Another useful graphical representation is the following:

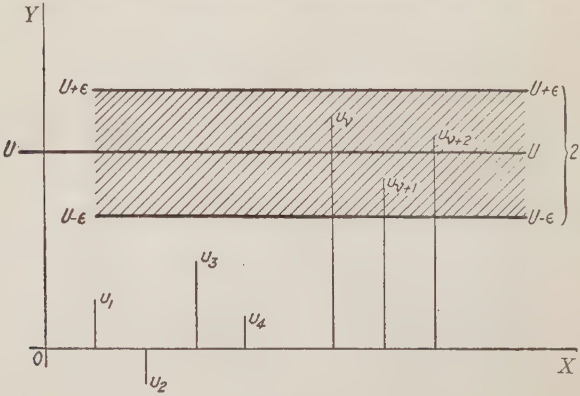

FIG. 3.

With two rectangular axes OX and OY as in Analytic Geometry, lay off the subscripts n as abscissas and the values u_n as ordinates, and also draw the line $y = U$ and the lines $y = U - \epsilon$ and $y = U + \epsilon$, at a distance ϵ on each side, giving a horizontal band of width 2ϵ. Then for each ϵ an index ν can be found such that all the points $u_n (n > \nu)$ fall within this ϵ-band.

22. Convergence, Divergence and Oscillation.—*A sequence which has a finite limit U is called* **convergent,** and we say that it converges to, or tends toward, or approaches the limit U.

We may put this definition in the form:

A sequence (u_n) is convergent when a finite number U exists such that, for any $\epsilon > 0$, however small, an index ν can be found such that for every $n > \nu$, we have $|U - u_n| < \epsilon$.

A sequence which has an infinite limit is called **divergent.** *If the limit is properly infinite, we say that the sequence is* **properly divergent,** *and if the limit is improperly infinite, we say that the sequence is* **improperly divergent.** *For every divergent sequence (u_n), we have $L|u_n| = +\infty$.*

A sequence which is neither convergent nor divergent is called **oscillating.** Such a sequence cannot have a limit, but its terms oscillate between certain values. Such a sequence can only occur among bounded sequences.

Examples.—(1) The sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$ is convergent and approaches the limit 1.

(2) The sequence $1, 3, 5, 7, \ldots$ is properly divergent, and $1, -3, 5, -7, \ldots$ is improperly divergent.

(3) The sequence $1, 0, 1, 0, 1, 0, \ldots$ is oscillating, oscillating between the upper and lower bounds 1 and 0. The sequence $0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \ldots$ oscillates between $+1$ and -1.

Some writers use the term "divergent" for all non-convergent sequences.

23. Remark on the Definition of Limit.—In the definition of limit, it is to be especially noted that the inequality

$|U - u_n| < \epsilon$ must be satisfied for *every* $n > \nu$. It is not sufficient to be able merely to find a value of n which will make $|U - u_n|$ as small as we please, but this inequality must be satisfied for *all subsequent* values of n. This is illustrated by taking the sequence $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{4}{5}$, . . . , in which, by taking n large enough, we can find a term which is as small as we please, but this is not true for all following values of n, odd and even; in fact, the sequence oscillates between 0 and 1, and has no limit.

24. Theorem.—*If the elements of the sequence* (u_n) *are complex numbers* $u_n = a_n + ib_n$, *the necessary and sufficient condition for the convergence of the complex sequence* (u_n) *is that the two real sequences* (a_n) *and* (b_n) *both converge; and if* $La_n = A$ *and* $Lb_n = B$, *then* $Lu_n = A + iB$.

For, if the complex sequence (u_n) is convergent, with limit $A + iB$, we have by definition, for every $\epsilon > 0$, however small,

$$|(A + iB) - (a_n + ib_n)| < \epsilon$$

Def of conv.

for every $n > \nu$. But since

$$|(A + iB) - (a_n + ib_n)| = \sqrt{(A - a_n)^2 + (B - b_n)^2},$$

we have

$$\sqrt{(A - a_n)^2 + (B - b_n)^2} < \epsilon, \quad n > \nu.$$

Neither $|A - a_n|$ nor $|B - b_n|$ can be greater than

$$\sqrt{(A - a_n)^2 + (B - b_n)^2},$$

so that

$$|A - a_n| < \epsilon, \quad |B - b_n| < \epsilon, \quad n > \nu,$$

and it follows from this that both sequences (a_n) and (b_n) converge with limits A and B respectively. Thus the condition is necessary.

If, conversely, the sequences (a_n) and (b_n) are both convergent with limits A and B respectively, we have

$$\epsilon > 0, \nu, |A - a_n| < \frac{\epsilon}{2}, |B - b_n| < \frac{\epsilon}{2}, \quad n > \nu,$$

and therefore

$$|(A + iB) - (a_n + ib_n)| \leqq |A - a_n| + |B - b_n| < \epsilon,$$
$$n > \nu.$$

Hence, the sequence $(a_n + ib_n)$ converges to the limit $A + iB$, and the condition is sufficient.

25. Theorem.—*If the real sequence* (u_n) *is improperly divergent, its positive and negative terms each form properly divergent sequences whose limits are respectively* $+\infty$ and $-\infty$.

For, let (a_n) be the sequence of positive terms of (u_n) and (b_n) the sequence of negative terms. Then we have

$$G > 0,\ \nu,\ |u_n| > G,\ n > \nu,$$

however large G may be; hence an index ν' can be found such that

$$a_n > G,\ b_n < -G,$$

for every $n > \nu'$, and hence

$$La_n = +\infty,\ Lb_n = -\infty.$$

26. General Principle of Convergence.—The application of the general *definition* of convergence to the terms of a given sequence involves a knowledge of the limit U; and therefore it is not adapted to use in general. The following important theorem, called[1] the **general principle of convergence,** gives a criterion for the existence of a limit which is better adapted for practical use since it does not involve a knowledge of the limit itself, but only of the terms of the sequence.

Theorem.—*A necessary and sufficient condition for the convergence of a sequence* (u_n) *is that, for any* $\epsilon > 0$, *as small as we please, a finite index* ν *exists such that* $|u_{n+p} - u_n| < \epsilon$ *for every* $n > \nu$, *and for every value of* p.

To prove this condition necessary, suppose that the sequence (u_n) converges to the limit U, then

$$\epsilon > 0,\ \nu,\ |U - u_n| < \frac{\epsilon}{2},\ n > \nu.$$

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[1] After P. du Bois-Reymond.

But

$$u_{n+p} - u_n \equiv (u_{n+p} - U) + (U - u_n),$$

$$\therefore |u_{n+p} - u_n| \leq |u_{n+p} - U| + |U - u_n|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for every $n > \nu$ and for every p, so that $|u_{n+p} - u_n| < \epsilon$ for every $n > \nu$ and for every p. Thus the condition is necessary.

To prove the condition sufficient, if we suppose first that the sequence has an infinite number of terms equal to one another, $= U$ say, then if $|u_{n+p} - u_n| < \epsilon$ for $n > \nu$, $p = 1, 2, 3, \ldots$, we may take $u_{n+p} = U$ for some value of p, and then we have $|U - u_n| < \epsilon$, $n > \nu$, so that in this case the sequence converges to the limit U.

If there are not an infinite number of terms of the sequence equal to one another, it contains an infinite number of distinct terms which form an infinite aggregate E. This aggregate is bounded. For, since $|u_{n+p} - u_n| < \epsilon$, $n > \nu$, $p = 1, 2, 3, \ldots$, we can write

$$u_n - \epsilon < u_{n+p} < u_n + \epsilon,$$

which shows that all the terms of the sequence, except the first ν, are contained in the interval $(u_\nu - \epsilon, u_\nu + \epsilon)$. Then, since the aggregate is bounded, it must have at least one limiting point within this interval $(u_\nu \pm \epsilon)$, by Art. **5.** There cannot be two such limiting points U and U', since the distance between these two points could be at most 2ϵ, and ϵ can be chosen as small as we please.

Hence, the aggregate E has one and only one limiting point, and by definition of limit, the sequence (u_n) has a limit and is convergent. Thus the condition is sufficient.

This proof applies only to real sequences. For complex sequences the proof follows directly from the result just demonstrated by applying Art. **24.**

This fundamental theorem was first given by Bolzano, and somewhat later was stated again by Cauchy and by

Abel. It was for a long time accepted as self-evident; and it was only rigorously proved when a systematic theory of irrational numbers was built up.

27. Monotonic Sequences.—*A real sequence* (u_n) *each of whose terms is greater than the preceding, that is, such that* $u_n < u_{n+1}$ *for every* n, *is called an* **increasing** *sequence. If each term is less than the preceding, that is,* $u_n > u_{n+1}$ *for every* n, *the sequence is called a* **decreasing** *sequence. In either case it is called* **univariant.**

If $u_n \leqq u_{n+1}$ for every n, the sequence is called **monotonic increasing.** If $u_n \geqq u_{n+1}$ for every n, the sequence is called **monotonic decreasing.** In either case it is called **monotonic.**

Examples.—(1) The sequence $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ is increasing, (2) the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ is decreasing, (3) the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{5}{6}, \ldots$ is monotonic increasing, and (4) the sequence $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \ldots$ is monotonic decreasing.

28. Theorem.—*A bounded monotonic sequence is always convergent.*

In the proof, we can evidently restrict ourselves to the case of a monotonic increasing sequence, for if we change the signs of the terms of a monotonic decreasing sequence, we obtain a monotonic increasing sequence.

Suppose then that the sequence (u_n) is monotonic increasing and bounded above, then, since it is bounded, it has an upper bound, $= U$ say. By definition of upper bound, there is a term of the sequence, say u_ν, which is $> U - \epsilon$, however small $\epsilon > 0$, since the sequence evidently cannot have a greatest term. Then, since the sequence is monotonic increasing, all the terms after u_ν must be $\geqq U - \epsilon$ and $\leqq U$, hence lie in the interval $(U - \epsilon, U)$.

$$\therefore |U - u_n| < \epsilon, n > \nu,$$

and the sequence is convergent with limit U.

29. Theorem.—*An unbounded monotonic sequence is properly divergent.*

As before, consider the case of the monotonic increasing sequence. This sequence cannot be oscillating or convergent since it is unbounded, and it cannot be improperly divergent since the terms of the sequence must ultimately be all positive, hence the sequence must be properly divergent.

30. Limit, Extreme Limits and Bounds.—By Art. **15,** the upper limit of a sequence will coincide with the upper bound except when the upper bound is also the greatest term of the sequence. Similarly for lower limit and lower bound.

If the upper and lower limits of a sequence are equal, $\overline{L} = \underline{L}$, then this common value, U say, is *the* limit of the sequence. For, in this case, for every $n > n_0$ we have $u_n < U + \epsilon$ and also $u_n > U - \epsilon$ (Art. **18**), and hence $U - \epsilon < u_n < U + \epsilon$, $n > n_0$, so that U is the limit of (u_n). Hence, if $\overline{L}u_n = \underline{L}u_n = U$, then $U = Lu_n$, and (u_n) is convergent.

If the upper and lower limits of a bounded sequence are not equal, then the sequence is oscillating.

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CHAPTER III

THEORY OF LIMITS

31. Operations with Limits. Theorem.—*If* $Lu_n = U$ *and* $Lv_n = V$, *then*

$$L(u_n \pm v_n) = U \pm V.$$

For,

$$|(U \pm V) - (u_n \pm v_n)| = |(U - u_n) \pm (V - v_n)|$$
$$\leqq |U - u_n| + |V - v_n|;$$

and since $u_n \to U$, we have

$$(a) \qquad \epsilon > 0,\ \nu_1,\ |U - u_n| < \frac{\epsilon}{2},\ n > \nu_1,$$

and since $v_n \to V$, we have

$$(b) \qquad \epsilon > 0,\ \nu_2,\ |V - v_n| < \frac{\epsilon}{2},\ n > \nu_2,$$

and if we choose $\nu > \nu_1, \nu_2$, we get by adding (a) and (b),

$$|(U \pm V) - (u_n \pm v_n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\ n > \nu,$$

which proves our theorem.

32. Theorem.—*If* $Lu_n = U$ *and* $Lv_n = V$, *then*

$$L(u_n \cdot v_n) = U \cdot V.$$

For,

$$UV - u_nv_n = U(V - v_n) + v_n(U - u_n),$$

and

$$(a) \qquad |UV - u_nv_n| \leqq |U| \cdot |V - v_n| + |v_n| \cdot |U - u_n|.$$

Since $v_n \to V$, we have

$$(b) \qquad |v_n| < g,\ n > \nu_1,$$

where g is some finite constant, for otherwise the sequence would be unbounded and could not have a finite limit. We have also, since $v_n \to V$,

$$(c) \qquad |V - v_n| < \frac{\epsilon}{2|U|},\ n > \nu_2,$$

and since $u_n \to U$,

$$(d) \qquad |U - u_n| < \frac{\epsilon}{2g},\ n > \nu_3.$$

By taking $\nu > \nu_1, \nu_2, \nu_3$, we can use this ν for all three inequalities $(b) - (d)$. Then (a) gives

$$|UV - u_n v_n| < |U| \cdot \frac{\epsilon}{2|U|} + g \cdot \frac{\epsilon}{2g} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\ n > \nu,$$

which proves the theorem.

33. Theorem.—*If* $Lu_n = U$ *and* $Lv_n = V$, *and if* $v_n \neq 0$ *and* $V \neq 0$, *then*

$$L\left(\frac{u_n}{v_n}\right) = \frac{U}{V}.$$

For, we can write

$$\frac{U}{V} - \frac{u_n}{v_n} = \frac{Uv_n - Vu_n}{V \cdot v_n} = \frac{(Uv_n - UV) + (UV - Vu_n)}{V \cdot v_n}$$

$$= \frac{U(v_n - V)}{V \cdot v_n} + \frac{U - u_n}{v_n}.$$

$$(a) \qquad \therefore \left|\frac{U}{V} - \frac{u_n}{v_n}\right| \leq \frac{|U|}{|V| \cdot |v_n|} \cdot |v_n - V| + \frac{|U - u_n|}{|v_n|}$$

Since $V \neq 0$, we have $|V| > 0$. Let h be a real number such that $0 < h < |V|$. Then an index ν' can be found such that

$$(b) \qquad |v_n| > h \quad \text{for} \quad n > \nu',$$

since $v_n \neq 0$. Since $v_n \to V$ and $u_n \to U$, we have

$$(c) \qquad \epsilon > 0,\ \nu_1,\ |V - v_n| < \frac{\epsilon |V| h}{2|U|},\ n > \nu_1,$$

$$(d) \qquad \epsilon > 0,\ \nu_2,\ |U - u_n| < \frac{\epsilon h}{2},\ n > \nu_2.$$

Taking $\nu > \nu_1, \nu_2, \nu'$, we get from (a) – (d):

$$\left|\frac{U}{V} - \frac{u_n}{v_n}\right| < \frac{|U|}{|V| \cdot h} \cdot \frac{\epsilon |V| \cdot h}{2\,|U|} + \frac{1}{h} \cdot \frac{\epsilon h}{2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\ n > \nu,$$

from which our theorem follows.

34. Fundamental Limit Theorem.—The results of Arts. **31–33** may be summarized and written:

$$(a) \qquad L(u_n \pm v_n) = Lu_n \pm Lv_n,$$

$$(b) \qquad L(u_n \cdot v_n) = Lu_n \cdot Lv_n,$$

if the sequences (u_n) and (v_n) are convergent, and

$$(c) \qquad L\left(\frac{u_n}{v_n}\right) = \frac{Lu_n}{Lv_n},$$

if the sequences (u_n) and (v_n) are convergent and if $Lv_n \neq 0$ and every $v_n \neq 0$.

By combination of these results, we arrive at the general

Theorem.—*If f denotes any finite combination of the four elementary (rational) operations, i. e., if f is any rational function, and (u_n), (v_n), (w_n), . . . are convergent sequences, and if no zero divisor occurs in f, then*

$$Lf(u_n, v_n, w_n, \ . \ . \ . \ ) = f(Lu_n, Lv_n, Lw_n, \ . \ . \ . \ )$$

35. Limit of Quotient.—If the sequence (u_r) is convergent (with a finite limit) and $v_n \rightarrow 0$, we cannot infer that $L\,\dfrac{u_n}{v_n} = \infty$ without first showing that $\dfrac{u_n}{v_n}$ has a fixed sign.

This may be illustrated by taking $u_n = 1$ and $v_n = (-1)^n \cdot \dfrac{1}{n}$, so that $v_n \rightarrow 0$, but $\dfrac{u_n}{v_n} = (-1)^n \cdot n$, and hence $\dfrac{u_n}{v_n}$ oscillates between $+\infty$ and $-\infty$, and so is improperly divergent.

If $u_n \rightarrow 0$ and $v_n \rightarrow 0$, the quotient $\dfrac{u_n}{v_n}$ may or may not have a limit.

This is illustrated by taking $u_n = \frac{1}{n}$, $v_n = (-1)^n \cdot \frac{1}{n}$, so that $\frac{u_n}{v_n} = (-1)^n$, which oscillates.

If (u_n) diverges, say $u_n \to +\infty$, and (v_n) converges, say to a positive limit, it is easy to see that

$$(u_n \pm v_n) \to \infty,\ u_nv_n \to \infty,\ \frac{u_n}{v_n} \to \infty,\ \frac{v_n}{u_n} \to 0;$$

the only case of exception arising when $v_n \to 0$, and then the sequences u_nv_n and $\frac{u_n}{v_n}$ both need special discussion.

If both (u_n) and (v_n) diverge to ∞, we have

$$(u_n + v_n) \to \infty,\ (u_nv_n) \to \infty,$$

but both $(u_n - v_n)$ and $\left(\frac{u_n}{v_n}\right)$ have to be specially examined.

36. Theorem.—*If sequences (u_n) and (v_n) are convergent, and if $Lu_n > Lv_n$, then an index ν exists such that $u_n > v_n$ for every $n > \nu$. Conversely, if $u_n < v_n$ for every n (or for every $n >$ some ν), and if the sequences (u_n) and (v_n) are convergent, then $Lu_n \leqq Lv_n$.*

For, if $Lu_n > Lv_n$, we can find a number k between these limits, and if $Lu_n > k$, we can find n sufficiently large, say $n > \nu_1$, such that $u_n > k$, and similarly if $Lv_n < k$, we can find ν_2 such that $v_n < k$ for every $n > \nu_2$, and hence we have $u_n > v_n$ for $n > \nu$, where $\nu > \nu_1, \nu_2$.

But if when $u_n < v_n$, we should assume that $Lu_n > Lv_n$, then it would follow from the preceding that $u_n > v_n$, contrary to hypothesis.

One might be tempted to conclude from $u_n < v_n$ that $Lu_n < Lv_n$, but the difference $v_n - u_n$, although always positive, may $\to 0$, so that the only correct conclusion is $Lu_n \leqq Lv_n$.

37. Theorem.—*If $u_n \geqq v_n$ for every n, and if $Lv_n = +\infty$, then $Lu_n = +\infty$.*

For, by hypothesis, any large

$$G > 0,\ \nu,\ v_n > G,\ n > \nu,$$

and since $u_n \geqq v_n$, we have

$$G > 0,\ \nu,\ u_n > G,\ n > \nu,$$

and therefore $u_n \to +\infty$.

38. Cauchy-Stolz Theorem.—*If (u_n) is any real sequence, and if (v_n) is an increasing positive sequence with the limit $+\infty$, and if the limit*

$$(a) \qquad L\,\frac{u_{n+1} - u_n}{v_{n+1} - v_n} = l$$

exists, finite or infinite, then the limit $L\,\dfrac{u_n}{v_n}$ exists $= l$:

$$L\,\frac{u_n}{v_n} = L\,\frac{u_{n+1} - u_n}{v_{n+1} - v_n}.$$

1° : l finite.

If we write

$$\epsilon > 0,\ \nu,\ l - \epsilon < \frac{u_{i+1} - u_i}{v_{i+1} - v_i} < l + \epsilon,\ i > \nu,$$

since $(v_{i+1} - v_i)$ is positive, we have

$$(l - \epsilon)(v_{i+1} - v_i) < u_{i+1} - u_i < (l + \epsilon)(v_{i+1} - v_i),$$

and substituting in place of i the values $n - 1,\ n - 2,\ \ldots,\ m + 1,\ m\ (m \geqq \nu)$, and adding, we get

$$(l - \epsilon)(v_n - v_m) < u_n - u_m < (l + \epsilon)(v_n - v_m),$$

and since $v_n > 0$, giv

$$(l - \epsilon)\left(1 - \frac{v_m}{v_n}\right) + \frac{u_m}{v_n} < \frac{u_n}{v_n} < (l + \epsilon)\left(1 - \frac{v_m}{v_n}\right) + \frac{u_m}{v_n}.$$

Since $v_n \to +\infty$, we get by keeping m fixed and letting $n \to \infty$,

$$\underset{n}{L}\left\{\left(l - \epsilon\right)\left(1 - \frac{v_m}{v_n}\right) + \frac{u_m}{v_n}\right\} = l - \epsilon,$$

$$\underset{n}{L}\left\{\left(l + \epsilon\right)\left(1 - \frac{v_m}{v_n}\right) + \frac{u_m}{v_n}\right\} = l + \epsilon.$$

Hence,

$$l - \epsilon \leqq \underline{L}_n \frac{u_n}{v_n} \leqq \overline{L}_n \frac{u_n}{v_n} \leqq l + \epsilon.$$

But these upper and lower limits are independent of m and of ϵ, so that the inequalities can only be true if each of these extreme limits is equal to l:

$$\underline{L}_n \frac{u_n}{v_n} = l = \overline{L}_n \frac{u_n}{v_n},$$

so that we have

$$L_n \frac{u_n}{v_n} = l.$$

2° : l infinite, $l = +\infty$.

We have here, however great $G > 0$ may be,

$$\frac{u_{n+1} - u_n}{v_{n+1} - v_n} > G,\ n > \nu,$$

or

$$u_{n+1} - u_n > G(v_{n+1} - v_n).$$

Substituting for n the values $m, m + 1, \ldots, m + p - 1$ $(m > \nu)$, in succession, and adding, we get

$$u_{m+p} - u_m > G(v_{m+p} - v_m).$$

$$\therefore \frac{u_{m+p}}{v_{m+p}} > \frac{u_m}{v_{m+p}} + G\left(1 - \frac{v_m}{v_{m+p}}\right).$$

As $p \to \infty$, with m fixed, the limit of the right-hand side is G, so that

$$\underline{L}_p \frac{u_{m+p}}{v_{m+p}} \geqq G, \quad >$$

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and since this lower limit is independent of m and of G, we have

$$L_n \frac{u_n}{v_n} = +\infty.$$

39. Theorem.—*If* $Lu_n = U$ *(finite or infinite), then*

$$L \frac{1}{n}(u_1 + u_2 + \ldots + u_n) = U.$$

For, if we put

$$U_n = u_1 + u_2 + \quad . \quad . \quad . \quad + u_n,$$

then $U_n - U_{n-1} = u_n$, and we may write

$$u_n = \frac{U_n - U_{n-1}}{n - (n - 1)}.$$

By the Cauchy-Stolz theorem, we then have

$$L \frac{U_n}{n} = U,$$

or

$$L \frac{1}{n}(u_1 + u_2 + \quad . \quad . \quad . \quad + u_n) = U = Lu_n.$$

40. Cesàro's Theorem.—*If* $Lu_n = U$ *and* $Lv_n = V$, *then*

$$L \frac{1}{n}(u_1v_n + u_2v_{n-1} + \quad . \quad . \quad . \quad + u_nv_1) = UV.$$

Since $Lu_n = U$ and $Lv_n = V$, we can find a positive number A such that for every n,

(*a*) $$|u_n| < A, |v_n| < A.$$

By Art. **32,** $Lu_nv_n = UV$, so that

(*b*) $$\epsilon > 0,\ p',\ |u_nv_n - UV| < \frac{\epsilon}{2},\ n > p'.$$

Now

$$\begin{aligned} |u_\alpha v_\beta - UV| &\leqq |u_\alpha v_\beta - u_\alpha v_\alpha| + |u_\alpha v_\alpha - UV| \\ &\leqq |u_\alpha| \cdot |v_\beta - v_\alpha| + |u_\alpha v_\alpha - UV|. \end{aligned}$$

By (*a*), $|u_\alpha| < A$ for every α. Since $v_n \rightarrow V$, by the general principle of convergence (Art. **26**), we can find p'' such that for $\beta > p''$, $\alpha > p''$, we have

(*c*) $$|v_\beta - v_\alpha| < \frac{\epsilon}{2A}.$$

By (*b*),

$$|u_\alpha v_\alpha - UV| < \frac{\epsilon}{2},\ \alpha > p'.$$

Hence,

(*d*) $$|u_\alpha v_\beta - UV| < A \cdot \frac{\epsilon}{2A} + \frac{\epsilon}{2} < \epsilon$$

for $\alpha, \beta > p$, where $p > p', p''$.

(*e*) $$u_\alpha v_\beta = UV + \theta\epsilon,$$

where $|\theta| < 1$, and $\alpha, \beta > p$.

Now let us choose $n > 2p$, and divide the sum

$$u_1v_n + u_2v_{n-1} + \dots + u_{n-1}v_2 + u_nv_1$$

into three parts:

$$\begin{aligned} &u_1v_n + u_2v_{n-1} + \dots + u_pv_{n-p+1} && (p \text{ terms}),\\ &u_{p+1}v_{n-p} + \dots + u_{n-p}v_{p+1} && (n - 2p \text{ terms}),\\ &u_{n-p+1}v_p + \dots + u_nv_1 && (p \text{ terms}). \end{aligned}$$

The sum of the first and third parts is, in absolute value, $<2pA^2$, since each term is $<A^2$ in absolute value by (*a*), and there are $2p$ such terms; we can then represent their sum by

(*f*) $$2\theta' pA^2, \ |\theta'| < 1.$$

For the second part, we have by (*e*),

(*g*) $$\sum_{\substack{\alpha=p+1\\ \beta=n-p}}^{\substack{\beta=p+1\\ \alpha=n-p}} (u_\alpha v_\beta) = u_{p+1}v_{n-p} + \dots + u_{n-p}v_{p+1} = (n-2p)(UV + \theta\epsilon).$$

Combining (*f*) and (*g*), we get

$$\frac{1}{n}(u_1v_n + \dots + u_nv_1) = \frac{2p\theta' A^2}{n} + \left(1 - \frac{2p}{n}\right)(UV + \theta\epsilon).$$

Keeping p fixed and letting $n \to \infty$, A being a constant and ϵ being arbitrarily small, we obtain

$$L\,\frac{1}{n}(u_1v_n + \dots + u_nv_1) = UV.$$ [1]

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[1] Borel, Leçons sur les séries divergentes, pp. 88–90.

41. Abel's Identity.—*If* $U_n = \sum_1^n u_i$, *then*

$$\sum_{n=1}^{p} u_n v_n = \sum_{n=1}^{p-1} U_n(v_n - v_{n+1}) + U_p v_p.$$

For, since $u_n = U_n - U_{n-1}$, we have

$$\begin{aligned}\sum_{n=1}^{p} u_n v_n &= U_1 v_1 + (U_2 - U_1)v_2 + (U_3 - U_2)\cdot v_3 + \cdots \\ &\qquad + (U_p - U_{p-1})v_p \\ &= U_1(v_1 - v_2) + U_2(v_2 - v_3) + \cdots \\ &\qquad + U_{p-1}(v_{p-1} - v_p) + U_p v_p \\ &= \sum_{n=1}^{p-1} U_n(v_n - v_{n+1}) + U_p v_p.\end{aligned}$$

This formula is sometimes also called **Abel's partial summation formula.**

42. Abel's Lemma.—*If* (u_n) *is any sequence, and* B_p *is the upper bound (Max) of* $|U_1|, |U_2|, \ldots, |U_p|$, *then*

1° *If* (v_n) *is a positive monotonic decreasing sequence,*

$$\left|\sum_{n=1}^{p} u_n v_n\right| < B_p v_1;$$

2° *If* (v_n) *is a positive monotonic increasing sequence,*

$$\left|\sum_{n=1}^{p} u_n v_n\right| < 2B_p v_p.$$

For, if (v_n) be monotonic decreasing, $(v_n - v_{n+1})$ is positive, and we have by Abel's identity,

$$\begin{aligned}\left|\sum_{n=1}^{p} u_n v_n\right| &= \left|\sum_{n=1}^{p-1} U_n(v_n - v_{n+1}) + U_p v_p\right| \\ &\leq \sum_{n=1}^{p-1} |U_n|\cdot(v_n - v_{n+1}) + |U_p|\cdot v_p \\ &< B_p\left\{\sum_{n=1}^{p-1}(v_n - v_{n+1}) + v_p\right\} \\ &< B_p v_1,\end{aligned}$$

since

$$\sum_{n=1}^{p-1}(v_n - v_{n+1}) + v_p = (v_1 - v_2) + (v_2 - v_3) + \ . \ . \ . + (v_{p-1} - v_p) + v_p = v_1.$$

If (v_n) be monotonic increasing, $(v_n - v_{n+1})$ is negative, and we have by Abel's identity,

$$\left|\sum_{n=1}^{p} u_n v_n\right| \leqq \sum_{n=1}^{p-1} |U_n| \cdot |v_n - v_{n+1}| + |U_p| \cdot v_p$$
$$< B_p \left\{ \sum_{n=1}^{p-1} (v_{n+1} - v_n) + v_p \right\}$$
$$< 2B_p v_p,$$

since $\therefore < 2B_p v_p$

$$\sum_{n=1}^{p-1}(v_{n+1} - v_n) + v_p = (v_2 - v_1) + (v_3 - v_2) + \ . \ . \ . \ . + (v_p - v_{p-1}) + v_p = 2v_p - v_1.$$

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CHAPTER IV

SPECIAL LIMITS

43. Lr^n.—If $0 < r < 1$, then $r^n = r^{n-1} \cdot r$, so that r^n decreases as n increases; hence by Art. **28** r^n must approach a limit as $n \to \infty$,—call this limit $l \geqq 0$. If we now let $n \to \infty$ in the identity $r^n = r^{n-1} \cdot r$, we get $l = lr$, or $l(1 - r) = 0$, or $l = 0$.

If $0 > r > -1$, we have $0 < |r| < 1$, and therefore $|r|^n \to 0$, and hence $r^n \to 0$.

If $r > 1$, we have $\frac{1}{r} < 1$, then by the preceding, $L\frac{1}{r^n} = 0$ and $\frac{1}{r^n} < \epsilon$ if $n >$ some index ν, so that $r^n > \frac{1}{\epsilon}$ for $n > \nu$, and since $\frac{1}{\epsilon}$ can be taken as large as we please, we have $r^n \to +\infty$.

If $r < -1$, r^n oscillates between positive and negative values as n increases, and $|r|^n \to +\infty$ so that the sequence (r^n) is improperly divergent.

If r is complex, we have by the preceding, if $|r| < 1$, $|r|^n \to 0$, and if $|r| > 1$, $|r|^n \to +\infty$.

We have then

$$\underset{n}{L} r^n = \begin{cases} 0 \text{ if } |r| < 1, \\ \infty \text{ if } |r| > 1. \end{cases}$$

44. $\underset{n}{L} a_n^m$ when $a_n \to a$.—If m is a positive integer, by repetition of Art. **32,** we see that a_n^m, regarded as the product of m variables (all equal), must $\to a^m$. If $m = \frac{1}{m'}$, where m' is a positive integer, and we put $a_n^m = b_n$, we have $a_n = b_n^{m'}$. Now by what we have just proved, if $b_n \to b$ as $n \to \infty$, since m' is a positive integer, we have $b_n^{m'} \to b^{m'}$, and therefore $a_n \to b^{m'}$. Now since $a_n \to b^{m'}$ when $b_n \to b$, we can assert conversely that when $a_n \to a$, Lb_n must exist and we have $a = b^{m'}$ or $b = a^m$. Hence $La_n^m = a^m$ when m is the reciprocal of a positive integer. If $m = \frac{p}{q}$, where p and q are positive integers, we have, making use of the result just established,

$$La_n^{\frac{p}{q}} = L\left(a_n^{\frac{1}{q}}\right)^p = \left(La_n^{\frac{1}{q}}\right)^p = \left(a^{\frac{1}{q}}\right)^p = a^{\frac{p}{q}},$$

so that $La_n^m = a^m$ when m is any fraction. Finally, if m is a negative rational number, $m = -m'$, m' a positive fraction or integer, we have

$$La_n^m = L\frac{1}{a_n^{m'}} = \frac{1}{a^{m'}} = a^{-m'} = a^m.$$

Hence, *for every rational value of* m,

$$\underset{n}{L}a_n^m = (\underset{n}{L}a_n)^m.$$

45. $\mathbf{Lb^{a_n}}$ when $\mathbf{a_n \to a}$.—Assume first that $b < 1$. Let $\alpha_n = |a - a_n|$, then $\alpha_n \to 0$. For any $0 < \epsilon < 1$, we can find ν such that $(1 - \epsilon)^\nu < b$ for this and all larger values of ν (Art. **44**). Now keeping ν fixed, choose n so that $\alpha_n < \frac{1}{\nu}$ for this and all subsequent values of n. Then

$$1 - b^{\alpha_n} < 1 - (1 - \epsilon)^{\nu\alpha_n} < 1 - (1 - \epsilon) = \epsilon,$$

and hence if $1 - b^{\alpha_n} < \epsilon$, we have $Lb^{\alpha_n} = 1$, or since

$$b^{\alpha_n} = b^{a_n} \div b^a,$$

we have $Lb^{a_n} = b^a$.

If $b > 1$, we can write, since $\frac{1}{b} < 1$,

$$Lb^{a_n} = L\frac{1}{b^{-a_n}} = \frac{1}{b^{-a}} = b^a.$$

Hence, we have *for every* $b > 0$,

$$Lb^{a_n} = b^{La_n}.$$

46. L log $\mathbf{a_n}$ when $\mathbf{a_n \rightarrow a}$.—If b is the base of the system of logarithms used, then if $\alpha_n = \log a_n$, we have $a_n = b^{\alpha_n}$. If $\alpha_n \rightarrow \alpha$, then $a_n \rightarrow b^\alpha$ (Art. **45**); conversely, if $a_n \rightarrow a$, then α_n must have a limit, and $a = b^\alpha$, or $\alpha = \log a$. Hence,

$$L \log a_n = \log La_n.$$

47. $\mathbf{La_n^{b_n}}$ when $\mathbf{a_n \rightarrow a}$ and $\mathbf{b_n \rightarrow b}$.—Put $a_n{}^{b_n} = c_n$, then $\log c_n = b_n \log a_n$. By Arts. **32, 46,** we get $\log c = b \log a$, or $c = a^b$. Hence,

a constant

$$La_n{}^{b_n} = (La_n)^{Lb_n}.$$

48. $\mathbf{L\dfrac{a^n}{n!}}$.—Suppose $a > 0$, then from the identity

$$\frac{a^n}{n!} = \frac{a^{n-1}}{(n-1)!} \cdot \frac{a}{n},$$

we see that $\dfrac{a^n}{n!}$ begins to decrease as soon as $n > a$, but it remains positive; hence, since the sequence $\left(\dfrac{a^n}{n!}\right)$ is bounded and decreasing, by Art. **28** it must approach a limit $l \geqq 0$. From the above identity, as $n \rightarrow \infty$, we get $l = l \cdot 0 = 0$ and hence

$$L\frac{a^n}{n!} = 0.$$

It is at once evident that this result holds also for $a < 0$.

49. $\mathbf{L\dfrac{a^n}{n}}$.—In the Cauchy-Stolz theorem, take $u_n = a^n$, $(a > 0)$, $v_n = n$, then

$$L\frac{u_{n+1} - u_n}{v_{n+1} - v_n} = L\left\{a^n(a-1)\right\} = 0 \text{ or } \infty,$$

according as $a \leqq 1$ or $a > 1$. Hence,

$$L\frac{a^n}{n} = \begin{cases} 0 \text{ if } a \leqq 1, \\ \infty \text{ if } a > 1. \end{cases}$$

50. Cauchy's Theorem.—*If* $a_n > 0$, *and if* $L\ \dfrac{a_{n+1}}{a_n} = \rho \neq 0$, *then* $L\ \sqrt[n]{a_n} = \rho$.

Since $\dfrac{a_{n+1}}{a_n} \to \rho$, we may write

$$\rho - \epsilon < \frac{a_{n+1}}{a_n} < \rho + \epsilon$$

for $n > \nu$, however small ϵ may be. If $m \geqq \nu$,

$$\rho - \epsilon < \frac{a_{m+1}}{a_m} < \rho + \epsilon,$$

$$\rho - \epsilon < \frac{a_{m+2}}{a_{m+1}} < \rho + \epsilon,$$

$$\rho - \epsilon < \frac{a_{m+3}}{a_{m+2}} < \rho + \epsilon,$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\rho - \epsilon < \frac{a_{m+p}}{a_{m+p-1}} < \rho + \epsilon.$$

Multiplying these inequalities, we get

$$(\rho - \epsilon)^p < \frac{a_{m+p}}{a_m} < (\rho + \epsilon)^p,$$

and then

$$(a_m)^{\frac{1}{m+p}} (\rho - \epsilon)^{\frac{p}{m+p}} < \sqrt[m+p]{a_{m+p}} < (a_m)^{\frac{1}{m+p}} (\rho + \epsilon)^{\frac{p}{m+p}}.$$

Now let $p \to \infty$ while m is held fixed, then the two terms on the extreme right and left of this double inequality approach $\rho + \epsilon$ and $\rho - \epsilon$ respectively (Art. **47**), and hence for every $n > n_1 \geqq \nu$, we have

$$\rho - 2\epsilon < \sqrt[n]{a_n} < \rho + 2\epsilon,$$

so that

$$\sqrt[n]{a_n} \to \rho.$$

The converse of this theorem is not true. This is shown for example by the sequence

$$1,\ a,\ ab,\ a^2b,\ a^2b^2,\ .\ .\ .\ ,\ a^nb^{n-1},\ a^nb^n,\ .\ .\ .\ ,$$

where $a \neq b$. The ratio $\frac{a_{n+1}}{a_n}$ here is alternately a and b, while the radical $\sqrt[n]{a_n} = ab$ or $a\left(b^{\frac{n-1}{n}}\right)$, so that $\sqrt[n]{a_n} \rightarrow ab$ as $n \rightarrow \infty$ in either case, while the ratio approaches no limit.

51. $L\,\frac{\log n}{n}$.—In the Cauchy-Stolz theorem, take $u_n = \log n$, $v_n = n$, then

$$\frac{u_{n+1} - u_n}{v_{n+1} - v_n} = \log\left(1 + \frac{1}{n}\right) \rightarrow 0,$$
$$\therefore L\,\frac{\log n}{n} = 0.$$

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CHAPTER V

EXPONENTIAL AND LOGARITHMIC LIMITS AND INEQUALITIES

In this section we shall deal only with real numbers.

52. The Fundamental Exponential Limit: $\mathrm{L}\left(1+\frac{1}{\mathrm{n}}\right)^{\mathrm{n}}$.—We have identically,

$$\alpha^n - \beta^n = (\alpha - \beta)(\alpha^{n-1} + \alpha^{n-2}\beta + \ . \ . \ . \ + \alpha\beta^{n-2} + \beta^{n-1}).$$

In the second parenthesis, put α in place of β, and if we assume that $\alpha > \beta$, we get

$$\alpha^n - \beta^n < (\alpha - \beta)(\alpha^{n-1} + \alpha^{n-1} + \ . \ . \ . \ + \alpha^{n-1} \quad \text{to } n \text{ terms})$$

$-\beta^n < (\alpha-\beta)\, n \cdot \alpha^{n-1}$

$$< n(\alpha - \beta)\alpha^{n-1}.$$

$(\alpha - \beta)\, \alpha^{n-1}$

Transposing, we get

$$(a) \qquad \beta^n > \alpha^{n-1}\left\{\alpha - n(\alpha - \beta)\right\}.$$

Now substitute $\alpha = 1 + \frac{1}{n-1}$, $\beta = 1 + \frac{1}{n}$, and for $n \geq 2$, we obtain

$$\left(1 + \frac{1}{n}\right)^n > \left(1 + \frac{1}{n-1}\right)^{n-1},$$

or if we put $u_n = \left(1 + \frac{1}{n}\right)^n$,

$$u_n > u_{n-1},$$

so that the sequence (u_n) is increasing.

In the inequality (*a*) above, put $\alpha = 1 + \frac{1}{2m}$, $\beta = 1$, $n = m + 1$, and we have

$$1 > \frac{1}{2}\left(1 + \frac{1}{2m}\right)^m,$$

or squaring,

$$4 > \left(1 + \frac{1}{2m}\right)^{2m},$$

so that

$$u_{2m} < 4.$$

But we have already seen that $u_{2m-1} < u_{2m}$, so that $u_n < 4$ for every n. Hence, the sequence (u_n) is bounded.

We have now shown that the sequence (u_n) is bounded and increasing, hence by Art. **28** it must have a limit. This limit is of very great importance, and is denoted by the letter e:

$$(b) \qquad L\left(1 + \frac{1}{n}\right)^n = e.$$

On account of the great importance of this limit, we shall give another proof of its existence.

By the binomial theorem, for n a positive integer, we have

$$(c)\quad \left(1 + \frac{1}{n}\right)^n = 1 + 1 + \left(1 - \frac{1}{n}\right)\frac{1}{2!} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{1}{3!} + \cdots + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right)\frac{1}{n!}$$

Hence, for every n,

$$\left(1 + \frac{1}{n}\right)^n > 2.$$

If we replace each of the differences $1 - \frac{1}{n}$, $1 - \frac{2}{n}$, . . . , $1 - \frac{n-1}{n}$ by the greater number 1, we obtain

$$\left(1 + \frac{1}{n}\right)^n < 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!},$$

and since $r! > 2^{r-1}$, we have

$$\left(1+\frac{1}{n}\right)^n < 2 + \frac{1}{2} + \frac{1}{2^2} + \ . \ . \ . \ + \frac{1}{2^{n-1}},$$

and summing this geometric progression, we find

$$\left(1+\frac{1}{n}\right)^n < 3 - \frac{1}{2^{n-1}} < 3.$$

Hence, we have shown that $\left(1+\frac{1}{n}\right)^n$ is bounded.

Since all the terms of (c) are positive, if we throw away a part of the terms, we shall diminish the value of the expression, so that if $m < n$,

$$\left(1+\frac{1}{n}\right)^n > 2 + \left(1-\frac{1}{n}\right)\frac{1}{2!} + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{1}{3!} + \ . \ . \ . \\ + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \ . \ . \ . \ \left(1-\frac{m-1}{n}\right)\frac{1}{m!},$$

and since $m < n$, we have

$$1-\frac{1}{n} > 1-\frac{1}{m},\ 1-\frac{2}{n} > 1-\frac{2}{m},\ . \ . \ . \ ,\ 1-\frac{m-1}{n} > \\ 1-\frac{m-1}{m},$$

so that

$$\left(1+\frac{1}{n}\right)^n > 2 + \left(1-\frac{1}{m}\right)\frac{1}{2!} + \left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\frac{1}{3!} + \\ . \ . \ . \ + \left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right) \ . \ . \ . \ \left(1-\frac{m-1}{m}\right)\frac{1}{m!}.$$

The right-hand side is $\left(1+\frac{1}{m}\right)^m$ by (c), therefore

$$\left(1+\frac{1}{n}\right)^n > \left(1+\frac{1}{m}\right)^m,\ n > m,$$

hence, $\left(1+\frac{1}{n}\right)^n$ forms an increasing sequence.

Therefore, since $\left(1+\frac{1}{n}\right)^n$ is bounded and increasing, it must approach a limit, which must lie between 2 and 3.

53. $L\left(1+\frac{1}{a_n}\right)^{a_n}$ when $a_n \to \infty$.—If m is the greatest integer contained in a_n, we have $m \leqq a_n < m+1$, hence $m+1$ and therefore also m, increases to ∞ with n. We have

$$\left(1+\frac{1}{m+1}\right)^m < \left(1+\frac{1}{a_n}\right)^{a_n} < \left(1+\frac{1}{m}\right)^{m+1},$$

or

$$\frac{m+1}{m+2}\left(1+\frac{1}{m+1}\right)^{m+1} < \left(1+\frac{1}{a_n}\right)^{a_n} < \frac{m+1}{m}\left(1+\frac{1}{m}\right)^m.$$

Thus $\left(1+\frac{1}{a_n}\right)^{a_n}$ is included between two variables each of which $\to e$, hence

$$(a) \qquad L\left(1+\frac{1}{a_n}\right)^{a_n} = e.$$

The same result is obtained if $a_n \to -\infty$.

If we put $a_n = \frac{1}{\alpha}$, then $\alpha \to 0$ when $n \to \infty$, and we have

$$\underset{n}{L}\left(1+\frac{1}{a_n}\right)^{a_n} = \underset{\alpha \to 0}{L}\left(1+\alpha\right)^{\frac{1}{\alpha}},$$

hence

$$(b) \qquad \underset{\alpha \to 0}{L}\left(1+\alpha\right)^{\frac{1}{\alpha}} = e.$$

54. $L\left(1+\frac{a_n}{n}\right)^n$ when $a_n \to a$.—Put $n = a_n\alpha_n$; then if $a \neq 0$, we have $\alpha_n \to \infty$. Then

$$\left(1+\frac{a_n}{n}\right)^n = \left(1+\frac{1}{\alpha_n}\right)^{a_n\alpha_n} = \left[\left(1+\frac{1}{\alpha_n}\right)^{\alpha_n}\right]^{a_n},$$

and by Arts. **47, 53,** we get

$$L\left(1+\frac{a_n}{n}\right)^n = L\left[\left(1+\frac{1}{\alpha_n}\right)^{\alpha_n}\right]^{a_n} = \left[L\left(1+\frac{1}{\alpha_n}\right)^{\alpha_n}\right]^{La_n} = e^a.$$

Hence,

$$(a) \qquad L\left(1+\frac{a_n}{n}\right)^n = e^{La_n}.$$

As a special case, if x is a real constant, we have

$$(b) \qquad \underset{n}{L}\left(1+\frac{x}{n}\right)^n = e^x.$$

55. Exponential and Logarithmic Inequalities.—Since $e > \left(1+\frac{1}{n}\right)^n$, however large n may be, we have for $x > 0$,

$$(a) \qquad e^x - 1 > \left(1+\frac{1}{n}\right)^{nx} - 1.$$

Now if $\alpha > 1$,

$$\frac{\alpha^m - 1}{\alpha - 1} = \alpha^{m-1} + \alpha^{m-2} + \ . \ . \ . \ + \alpha + 1 > m,$$

$\alpha^{m}-1 > m(\alpha-1)$

$$(b) \qquad \therefore \alpha^m - 1 > m(\alpha - 1).$$

[illegible] if $\alpha < 1$

Taking $\alpha = 1 + \frac{1}{n}$, and $m = nx$, we get

$$\left(1+\frac{1}{n}\right)^{nx} - 1 > nx\left\{\left(1+\frac{1}{n}\right) - 1\right\}$$
$$> x,$$

hence by (a),

$$e^x - 1 > x, \text{ or } e^x > 1 + x.$$

Taking logarithms, we get from the last result,

$$\log_e (1 + x) < x.$$

Again, since $e < \left(1-\frac{1}{n}\right)^{-n}$ and $e^{-1} > \left(1-\frac{1}{n}\right)^n$, we have for $0 < x < 1$,

$$e^{-x} - 1 > \left(\frac{n-1}{n}\right)^{nx} - 1,$$

and putting $\alpha = \frac{n-1}{n}$, and $m = nx$ in (b), we find

$$\left(\frac{n-1}{n}\right)^{nx} - 1 > nx\left(\frac{n-1}{n} - 1\right) > -x.$$

$\left(1-\frac{1}{n}\right)^{nx} - 1 > -x$

Hence, $e^{-x} > 1 - x$, and $e^x < \frac{1}{1-x}\cdot$ From this we get

$$\log_e \frac{1}{1-x} > x, \text{ or } -\log_e (1-x) > x.$$

Summarizing, we have

$$(\alpha) \qquad \left\{\begin{array}{r} \left.\begin{array}{r} e^x > 1 + x, \\ \log (1+x) < x, \end{array}\right\} x > 0, \\ \left.\begin{array}{r} e^{-x} > 1 - x, \\ -\log (1-x) > x \end{array}\right\} 0 < x < 1. \end{array}\right.$$

Another method of proving these inequalities is as follows: For n a positive integer, we have by the binomial theorem,

$$\left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{1}{2!}\,\frac{n-1}{n}\,x^2 + \frac{1}{3!}\,\frac{(n-1)(n-2)}{n^2}\,x^3 + \cdots + \frac{1}{n!}\cdot\frac{(n-1)(n-2)\ \ldots\ 2.1}{n^{n-1}}\,x^n,$$

hence for $x > 0$,

$$\left(1 + \frac{x}{n}\right)^n \left\{\begin{array}{l} > 1 + x + \frac{1}{2}\frac{n-1}{n}\,x^2, \\ < 1 + x + \frac{1}{2}\,x^2 \cdot \frac{x^{n-1} - 1}{x - 1}. \end{array}\right.$$

If we now let $n \to \infty$, since $L\left(1 + \frac{x}{n}\right)^n = e^x$, we have for every positive x,

$$(d) \qquad e^x > 1 + x + \tfrac{1}{2}x^2 > 1 + x \ (0 < x \leqq \infty),$$

and for $x \leqq 1$,

$$(e) \quad e^x \leqq 1 + x - \frac{1}{2}\,\frac{x^2}{x-1} < 1 + x + \frac{x^2}{1-x} < \frac{1}{1-x} \qquad (0 \leqq x \leqq 1).$$

If we take reciprocals in these inequalities, we get

$$e^{-x} \left\{\begin{array}{ll} < \frac{1}{1+x} & (0 < x \leqq \infty), \\ > 1 - x & (0 < x < 1), \end{array}\right.$$

or if we put $-x$ in place of x:

$$(f)\qquad e^x\begin{cases} < \dfrac{1}{1-x} & (-\infty \leqq x < 0), \\ > 1 + x & (-1 \leqq x < 0). \end{cases}$$

By combination of the inequalities (d), (e), (f), we may write

$$(g)\qquad e^x\begin{cases} \geqq 1 + x & (-1 \leqq x \leqq \infty), \\ \leqq \dfrac{1}{1-x} & (-\infty \leqq x < 1). \end{cases}$$

56. $\underset{x\to 0}{L}\dfrac{\log(1+x)}{x}$.—We have

$$\underset{x\to 0}{L}\frac{\log(1+x)}{x} = \underset{x\to 0}{L}\log(1+x)^{\frac{1}{x}}$$

$$= \log \underset{x\to 0}{L}(1+x)^{\frac{1}{x}} = \log e = 1 \quad (\text{Art. } \mathbf{53}).$$

$$\therefore \underset{x\to 0}{L}\frac{\log(1+x)}{x} = 1.$$

57. $\underset{x\to 0}{L}\dfrac{a^x - 1}{x}$.—Put $a^x - 1 = y$, then $x = \log_a(1+y)$, and $y \to 0$ when $x \to 0$. Then

$$\underset{x\to 0}{L}\frac{a^x-1}{x} = \underset{y\to 0}{L}\frac{y}{\log_a(1+y)} = \underset{y\to 0}{L}\frac{1}{\log_a(1+y)^{\frac{1}{y}}}$$

$$= \frac{1}{\log_a\left\{\underset{y\to 0}{L}(1+y)^{\frac{1}{y}}\right\}} = \frac{1}{\log_a e} = \log_e a \ (\text{Art. } \mathbf{53}).$$

$$(a)\qquad \therefore \underset{x\to 0}{L}\frac{a^x-1}{x} = \log a.$$

58. Theorem.—*If $a > 1$, then*

$$(a)\qquad \underset{x}{L}\frac{a^x}{x} = \infty,\ \underset{x}{L}\frac{\log_a x}{x} = 0,\ \underset{x\to 0^+}{L} x\log_a x = 0.$$

In the Cauchy-Stolz theorem, take $u_n = a^x$, $v_n = x$, then

$$\frac{u_{n+1} - u_n}{v_{n+1} - v_n} = \frac{a^{x+1} - a^x}{x + 1 - x} = a^x(a - 1),$$

$$\frac{u_n}{v_n} = \frac{a^x}{x}.$$

Now $\underset{x}{L}\, a^x(a - 1) = \infty$ if $a > 1$, hence

(*b*)
$$\underset{x}{L}\, \frac{a^x}{x} = \infty.$$

The other two limits follow from this by transformation. Put $a^x = y$, then $x = \log_a y$, and $y \to \infty$ when $x \to \infty$. Then we have

$$\underset{y}{L}\, \frac{y}{\log_a y} = \underset{x}{L}\, \frac{a^x}{x} = \infty,$$

(*c*)
$$\therefore \underset{y}{L}\, \frac{\log_a y}{y} = 0.$$

Put $a^x = \frac{1}{y}$, then $x = -\log_a y$, and $y \to 0$ through positive values when $x \to \infty$, and

$$-\underset{y \to 0^+}{L}\, \frac{1}{y \log_a y} = \underset{x}{L}\, \frac{a^x}{x} = \infty,$$

(*d*)
$$\therefore \underset{y \to 0^+}{L}\, y \log_a y = 0.$$

59. Theorem.—*If* $a > 1$, and $k > 0$, then

(*a*)
$$\underset{x}{L}\, \frac{a^x}{x^k} = \infty,\ \underset{x}{L}\, \frac{\log_a x}{x^k} = 0,\ \underset{x \to 0^+}{L}\, x^k \log_a x = 0.$$

For we can write

$$\underset{x}{L}\, \frac{a^x}{x^k} = \underset{x}{L} \left\{ \frac{a^{\frac{x}{k}}}{x} \right\}^k = \left\{ \underset{x}{L} \left[\frac{\left(a^{\frac{1}{k}}\right)^x}{x} \right] \right\}^k.$$

Since $a > 1$ and $k > 0$, we have $a^{\frac{1}{k}} > 1$, and therefore by Art. **58**,

$$\underset{x}{L}\, \frac{\left(a^{\frac{1}{k}}\right)^x}{x} = \infty.$$

Hence,

$$\left\{ \underset{x}{L} \left[\frac{\left(a^{\frac{1}{k}}\right)^{x}}{x} \right] \right\}^{k} = \infty.$$

The other two results can be established in like manner if we put $y = \log_a x$ in one case, and $y = -\log_a x$ in the other case.

If k is negative, we see at once that

$$(b) \quad \underset{x}{L} \frac{a^x}{x^k} = \infty, \quad \underset{x}{L} \frac{\log_a x}{x^k} = \infty, \quad \underset{x \to 0^+}{L} x^k \log_a x = -\infty.$$

60. $\underset{n}{\mathbf{L}} \left\{ \frac{(\log n)^k}{n} \right\}$.—In the Cauchy-Stolz theorem, take $u_n = (\log n)^2$, $v_n = n$, then we have

$$\begin{aligned} \frac{u_{n+1} - u_n}{v_{n+1} - v_n} &= [\log (n+1)]^2 - [\log n]^2 \\ &= [\log (n+1) + \log n][\log (n+1) - \log n] \end{aligned}$$

$$= [\log (n+1) + \log n] \log \left(1 + \frac{1}{n}\right) < \frac{2}{n} \log (n+1).$$

(Art. **55**).

But

$$\frac{2}{n} \log (n+1) = \frac{2(n+1)}{n} \cdot \frac{\log (n+1)}{n+1} \to 0$$

by Art. **51**, hence we find

$$(a) \qquad \underset{n}{L} \frac{(\log n)^2}{n} = 0.$$

Similarly, we can prove that

$$(b) \qquad \underset{n}{L} \frac{(\log n)^k}{n} = 0,$$

where k is any positive integer.

61. $\underset{n}{\mathbf{L}} \sum_{i=1}^{n} \frac{1}{i}$.—Put

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ . \ . \ . \ + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i}.$$

By Art. **55**,

$$\log(1+i) - \log i = \log\left(1 + \frac{1}{i}\right) < \frac{1}{i}.$$

Put $i = 1, 2, 3, \ldots, n$ successively, and add the resulting inequalities, and we get

$$\log(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

log(n

or

(*a*) $$H_n > \log(n+1).$$

From this we see that, since $\log(n+1) \to \infty$, we have also

(*b*) $$\underset{n}{L} H_n = +\infty.$$

By the Cauchy-Stolz theorem, we have

$$L\frac{H_n}{\log n} = L\frac{H_n - H_{n-1}}{\log n - \log(n-1)} = L\frac{\dfrac{1}{n}}{\log\dfrac{n}{n-1}} =$$

$$L\frac{1}{\log\left(1-\dfrac{1}{n}\right)^{-n}} = \frac{1}{\log\left\{L\left(1-\dfrac{1}{n}\right)^{-n}\right\}} = \frac{1}{\log e} = 1.$$

(*c*) $$\therefore L\frac{H_n}{\log n} = 1.$$

By use of Art. **51,** we have

$$L\frac{H_n}{n} = L\cdot\frac{H_n}{\log n}\cdot\frac{\log n}{n} = L\cdot\frac{H_n}{\log n}\cdot L\frac{\log n}{n} = 0.$$

(*d*) $$\therefore L\frac{H_n}{n} = 0.$$

62. Euler's Constant.—Put $C_n \equiv H_n - \log n$. It is evident at once that C_n is always positive. Since

$$\frac{1}{n} < -\log\left(1-\frac{1}{n}\right) = \log\frac{n}{n-1},$$

we have

$$C_n - C_{n-1} = \frac{1}{n} - \log\frac{n}{n-1} < 0,$$

so that the sequence (C_n) is decreasing, and it is bounded below since C_n is always positive. Hence, by Art. **28,** C_n must → a finite limit. This limit is usually represented by C (sometimes by γ), and is called **Euler's constant:**

$$C = \underset{n}{L}(H_n - \log n).$$

Its value is

$$C = 0.577\,215\,664\,901\,532\ .\ .\ .$$

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CHAPTER VI

INFINITARY NOTATION

63. Orders of Infinites.—Suppose (u_n) and (v_n) are two sequences such that $Lu_n = +\infty$ and $Lv_n = +\infty$, and that $L\dfrac{u_n}{v_n}$ exists (finite or infinite).

If $L\dfrac{u_n}{v_n}$ is finite and $\neq 0$, we say that u_n *becomes infinite of the same order as* (or as fast as) v_n, or more briefly, **u_n is infinitarily equal to v_n.**

If $L\dfrac{u_n}{v_n} = 1$, we say **u_n is asymptotically equal to v_n,** and we denote this relation by $u_n \sim v_n$. Then if $L\dfrac{u_n}{v_n} = k$, we may write this relation $u_n \sim kv_n$.

If $L\dfrac{u_n}{v_n} = 0$, we say that u_n *becomes infinite of lower order* (or more slowly or more weakly) *than* v_n, or more briefly, **u_n is infinitarily less than v_n,** and we write $u_n \prec v_n$.

If $L\dfrac{u_n}{v_n} = \infty$, we say that u_n *becomes infinite of higher order* (or more rapidly or more strongly) *than* v_n, or more briefly, **u_n is infinitarily greater than v_n,** and we write $u_n \succ v_n$.

Examples.—(1) Since $L\dfrac{n^2+1}{n^2-1} = 1$, we have $n^2 + 1 \sim n^2 - 1$.

(2) We have $3n^3 - 2n^2 + 5 \sim \frac{3}{2}(2n^3 + 7)$, since $L\,\frac{3n^3 - 2n^2 + 5}{2n^3 + 7} = \frac{3}{2}$.

(3) $n + 2 \prec 2n^3 + 7$, since $L\,\frac{n+2}{2n^3 + 7} = 0$.

(4) $n^5 + 5 \succ n^2 + n + 1$, since $L\,\frac{n^5 + 5}{n^2 + n + 1} = \infty$

When $L\,\frac{u_n}{v_n} = 0$, *we sometimes write* $u_n = o(v_n)$. Thus $u_n \prec v_n$ and $u_n = o(v_n)$ are simply two different ways of expressing the same relation.

The preceding definitions may be extended to functions of a continuous variable. Suppose that $f_1(x)$ and $f_2(x)$ are two such functions, defined for all values of x greater than a given value x_0, which are positive, continuous, and steadily increasing, tending to ∞ with x. Then we write
$f_1(x) \sim f_2(x)$, $f_1(x) \sim k \cdot f_2(x)$, $f_1(x) \prec f_2(x)$, $f_1(x) \succ f_2(x)$,
according as

$$\underset{x}{L}\,\frac{f_1(x)}{f_2(x)} = 1,\ k,\ 0, \text{ or } \infty.$$

If $\underset{x}{L}\,\frac{f_1(x)}{f_2(x)} = 0$, we also write $f_1(x) = o[f_2(x)]$.

Examples.—(1) $x + \sin x \sim x$,
(2) $x + 1000\sqrt{x} \sim x$,
(3) $1 + \frac{1}{\log x} \sim 1$,
(4) $\frac{1}{x + \log x} \sim \frac{1}{x}$,
(5) $\sqrt{x} \sim \sqrt{x+1}$,
(6) $\sin x - x = o\,(x)$,
(7) $a^x \succ x\ (a > 1)$,
(8) $\log x \prec x$,
(9) $(1 + x)^m - 1 \sim mx$,
(10) $H_n = o\,(n)$.

64. Large O.—*If* (u_n) *and* (v_n) *are two sequences such that* $\left|\frac{u_n}{v_n}\right| < K$ *for every* $n >$ *some fixed value* n_0, *where* K *is a constant independent of* n, *we write*

$$u_n = O(v_n).$$

For example,

$$\frac{2n-1}{n^2+1} = O\left(\frac{1}{n}\right).$$

We may also express the above definition by saying that $u_n = O(v_n)$ means $\overline{L}\left|\frac{u_n}{v_n}\right| < \infty$. It is evident that if $L\left|\frac{u_n}{v_n}\right| = k$ (const.), then $u_n = O(v_n)$.

This symbol O may be also extended to the case of functions of a continuous variable. If $f(x)$ and $g(x)$ are defined for all positive values of $x > x_0$, and $g(x)$ is positive, and

$$\overline{L}_{x}\,\frac{|f(x)|}{g(x)} < \infty,$$

or if ξ and K exist such that for every $x \geqq \xi$, we have

$$|f(x)| < K \cdot g(x),$$

then we write

$$f(x) = O[g(x)].$$

Examples.—(1) $\sqrt{x} = O(x)$,
(2) $x + 1 = O(x)$,
(3) $x \sin x = O(x)$,
(4) $\log x = O(x^k)$.

We may denote by $O(1)$ any function of x which is defined for all values of x sufficiently large, and which either has a finite limit for $x \to \infty$, or at least for all sufficiently large values of x remains less in absolute value than some fixed bound.

Examples.—(5) $\sin x = O(1)$,

(6) $\dfrac{1+i}{x} = O(1)$,

(7) $\dfrac{\sin x}{x} = O(1)$,

(8) $\dfrac{\log x}{x} = O(1)$,

(9) $\dfrac{x}{x-10} = O(1)$,

(10) $e^{ix} = O(1)$.

65. Operations with O. Theorem I.—*From* $f_1(x) = O[g_1(x)]$ *and* $f_2(x) = O[g_2(x)]$, *it follows that*

$$f_1(x) + f_2(x) = O[g_1(x) + g_2(x)].$$

For, if $x \geqq \xi_1$,

$$|f_1(x)| < K_1 \cdot g_1(x),$$

and if $x \geqq \xi_2$,

$$|f_2(x)| < K_2 \cdot g_2(x),$$

then

$$\begin{aligned} |f_1(x) + f_2(x)| &\leqq |f_1(x)| + |f_2(x)| \\ &< K_1 g_1(x) + K_2 g_2(x) \\ &< K[g_1(x) + g_2(x)], \end{aligned}$$

for $x \geqq \xi$, where ξ is the greater of ξ_1, ξ_2, and K is the greater of K_1, K_2.

Theorem II.—*From* $f_1(x) = O[g_1(x)]$, *and* $f_2(x) = O\,[g_2(x)]$, *it follows that*

$$f_1(x) \cdot f_2(x) = O[g_1(x) \cdot g_2(x)].$$

For,

$$\begin{aligned} &|f_1(x)| < K_1 \cdot g_1(x) \text{ for } x \geqq \xi_1, \\ &|f_2(x)| < K_2 \cdot g_2(x) \text{ for } x \geqq \xi_2, \\ \therefore\ &|f_1(x).\ f_2(x)| < K_1 \cdot K_2 \cdot g_1(x) \cdot g_2(x) \text{ for } x \geqq \xi, \end{aligned}$$

where ξ is the greater of ξ_1, ξ_2.

Theorem III.—*If* $f_1(x) = O[g_1(x)]$, *and* $f_2(x) = o\,[g_2(x)]$, *then* $f_1(x) \,.\, f_2(x) = o[g_1(x) \,.\, g_2(x)]$.

For,

$$\overline{\underset{x}{L}}\, \frac{|f_1(x)|}{g_1(x)} = K < \infty,\ \underset{x}{L}\, \frac{f_2(x)}{g_2(x)} = 0,$$

therefore

$$\underset{x}{L}\, \frac{f_1(x)}{g_1(x)}\ \frac{f_2(x)}{g_2(x)} = 0.$$

66. Orders of Infinitesimals.—Let (u_n) and (v_n) be two sequences such that $Lu_n = 0$, $Lv_n = 0$, and suppose that $L\,\dfrac{u_n}{v_n}$ exists (finite or infinite).

Then we say that u_n *is an infinitesimal of higher order, the same order, or lower order, than* v_n, according as

$$L\,\frac{u_n}{v_n} = 0, \quad = k(\text{finite}) \neq 0, \text{ or } = \infty,$$

and we use the notation

$$u_n \prec v_n,\ u_n \sim v_n,\ u_n \succ v_n.$$

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CHAPTER VII

GENERAL PROPERTIES OF INFINITE SERIES

67. Infinite Series.—*If* $u_1, u_2, u_3, \dots, u_n, \dots$ *is an infinite sequence of real or complex terms, the symbol*

$$u_1 + u_2 + u_3 + \dots + u_n + \dots$$

is called an **infinite series.** An infinite series is regarded as *defined* when some law or rule is given by which the n^{th} term can be found.

An infinite series is usually given in one of the two following ways: The first few terms may be given, so that from these we can work out a formula for the n^{th} term or general term; or a formula for the n^{th} term may be given directly, and from this we may write down any particular term desired, or the first few terms. Often a combination of these two methods is used, the first few terms and also the n^{th} term being given.

Examples.—(1) For the series whose first few terms are

$$\frac{4}{1.3} + \frac{5}{2.4} + \frac{6}{3.5} + \dots,$$

the n^{th} term is $u_n = \dfrac{n+3}{n(n+2)}$.

(2) If the n^{th} term is $u_n = \dfrac{2n+1}{n^2+1}$, the series has for its first few terms, for $n = 1, 2, 3, \dots$,

$$3/2 + 5/5 + 7/10 + \dots$$

For the series $u_1 + u_2 + u_3 + \dots + u_n + \dots$ we often use the abbreviated notation $\sum u_n$, which is read "summation u_n," or $\sum_{n=1}^{\infty} u_n$, which is read "summation u_n from $n = 1$ to ∞."

68. Convergence, Divergence, Oscillation.—Put

$$U_n = u_1 + u_2 + u_3 + \ . \ . \ . + u_n = \sum_{i=1}^{n} u_i;$$

U_n is the sum of the first n terms of the series, and is called a **partial sum** of the series $\sum u_n$.

If the sequence (U_n) *is convergent and has the finite limit* $LU_n = U$, *we say that the series* $\sum u_n$ *is* **convergent** *and that* U *is its* **sum.**

By the definition of convergence of a sequence, we may write

$$\epsilon > 0, \ \nu, \ |U - U_n| < \epsilon, \ n > \nu.$$

If the sequence (U_n) *is divergent, we say that the series* $\sum u_n$ *is* **divergent,** *and it is* **properly divergent** *or* **improperly divergent** *according as the sequence* (U_n) *is properly or improperly divergent.*

If the series $\sum u_n$ is divergent, we have

$$G > 0, \ \nu, \ |U_n| > G, \ n > \nu.$$

If the sequence (U_n) *is oscillating, we say that the series* $\sum u_n$ *is* **oscillating.**

Some writers use the term "divergent" for all non-convergent series; but we prefer to distinguish between the various classes of non-convergent series by the above terminology.

It must be remembered that the "sum" U of a convergent series $\sum u_n$ as defined above, is a *limit*, and we must not assume without proof that the properties of a finite sum in Algebra will necessarily hold for such limits of sums as U.

Since convergent series are the only ones which have a *sum* as defined above, they are the only series which are used in the elementary theory of series, and indeed in earlier times it was thought that they were the only kind that could ever be used; but a modern theory of "divergent" (that is, non-convergent) series has been developed in recent years which enables us to make use of many classes of non-convergent series.

69. Notation.—Following Pierpont,[1] whenever practicable we shall adopt the following uniform notation. The terms of a series will be represented by small letters, the sum of the first n terms of a series $\sum u_n$ will be represented by U_n (the corresponding capital letter), the infinite series $\sum_{n+1}^{\infty} u_i$ by $\overline{U}_n$, and when no confusion can arise, we shall represent a series $\sum u_n$ and its sum by the same letter U. A sum of p consecutive terms after the n^{th} term will be designated by $U_{n,p} : U_{n,p} \equiv \sum_{n+1}^{n+p} u_i$.

70. Complex Series.—The definition of convergence applies to both real and complex series. But the conditions of convergence of a complex series may be expressed in terms of real series as follows. Let $\sum u_n$ be a series of complex terms, $u_n = a_n + ib_n$; then since

$$\sum_{m=1}^{n} u_m = \sum_{m=1}^{n} a_m + i \sum_{m=1}^{n} b_m,$$

it is evident that $\sum u_n$ will be convergent when and only when $\sum a_n$ and $\sum b_n$ are both convergent, and if A and B are the sums of these two real series, the sum of the complex series will be $U = A + iB$ (Art. **24**).

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[1] Pierpont, "Theory of Functions of Real Variables," vol. II, p. 77 (art. **79**).

[illegible]

71. Remainder.—The series

$$u_{n+1} + u_{n+2} + \ . \ . \ . \ + u_{n+p} + \ . \ . \ . \ = \sum_{n+1}^{\infty} u_i = \overline{U}_n$$

is called the **remainder after n terms** of the series $\sum u_n$.

The sum $U_{n,p} = u_{n+1} + u_{n+2} + \ . \ . \ . \ + u_{n+p}$ is called a **partial remainder** of the series.

Theorem.—*If* $\sum_1^\infty u_n$ *converges to the sum* U, *then* $\overline{U}_n = \sum_{p=1}^{\infty} u_{n+p}$ *converges and we have* $\overline{U}_n = U - U_n$, *and* $L\overline{U}_n = 0$. *If the series* $\overline{U}_n$ *converges, then* $\sum_1^\infty u_n$ *converges and we have* $U = U_n + \overline{U}_n$.

For we have identically, $U_{n+p} = U_n + U_{n,p}$, or $U_{n,p} = U_{n+p} - U_n$. Now if $\sum_1^\infty u_n$ is convergent, then $\underset{p}{L}U_{n+p} = U$, so that $\overline{U}_n = \underset{p}{L}U_{n,p} = U - U_n$, and hence the series $\overline{U}_n$ is convergent, and $\overline{U}_n = U - U_n$. In the equation $\overline{U}_n = U - U_n$, let $n \to \infty$, and we have $\underset{n}{L}\overline{U}_n = U - LU_n = U - U = 0$.

If $\overline{U}_n$ is convergent, then $\underset{p}{L}U_{n,p}$ exists $= \overline{U}_n$, and from $U_{n+p} = U_n + U_{n,p}$, we get by letting $p \to \infty$,

$$\underset{p}{L}U_{n+p} = U_n + \underset{p}{L}U_{n,p} = U_n + \overline{U}_n,$$

or

$$U = U_n + \overline{U}_n.$$

In the same way, we can prove the following

Theorem.—*If* $\sum u_n$ *is divergent or oscillating, so also is* $\overline{U}_n$, *and conversely, when* $\overline{U}_n$ *is divergent or oscillating, so also is* $\sum u_n$.

It is not in general possible to find the sum of a convergent series. The sum U_n of the first n terms of the series may be regarded as an approximation to the value of U, and then $\overline{U}_n$ is the error made in this approximation. Since $\overline{U}_n \rightarrow 0$ as $n \rightarrow \infty$ for a convergent series, this error may be made as small as we please by taking n large enough, that is, by adding up enough terms for our approximation. The series best adapted for use in numerical computation are those in which $\overline{U}_n \rightarrow 0$ rather rapidly.

If $\sum u_n$ is convergent with sum U, we may write

$$\epsilon > 0,\ \nu,\ |U - U_n| < \epsilon,\ n > \nu.$$

The least value of the index ν corresponding to a given ϵ measures in a way the rapidity of convergence of the series.

72. Theorem.—*In determining whether a given series is convergent or non-convergent, a finite number of its terms may be neglected; but if the series is convergent, its sum is thereby altered. The alteration of a finite number of terms of a series will not affect the convergence or non-convergence.*

This is evident at once since the sum of the neglected terms or of the altered terms will have a definite finite value.

73. Theorem.—*The convergence or the sum of a convergent series will not be changed if its terms are grouped without changing their order.*

This is evident from a consideration of the definition of limit. We may therefore introduce parentheses in a convergent series.

It is also evident that a properly divergent series remains properly divergent when its terms are grouped.

It is allowable to remove parentheses unless the resulting series is oscillating or divergent.

74. Theorem.—*If* $\sum u_n$ *is convergent with sum* U, *then* $\sum ku_n$ *is convergent with sum* kU, *if* k *is any finite constant. If* $\sum u_n$ *is non-convergent, so also is* $\sum ku_n$.

For the sum of n terms of $\sum ku_n$ is kU_n, and we have $LkU_n = k \cdot LU_n = kU$. If $LU_n = \pm\infty$, then $LkU_n = \pm\infty$, and if U_n has no limit, neither has kU_n.

75. The Geometric Series.—*A series*

$$1 + r + r^2 + r^3 + \dots + r^n + \dots = \sum_{n=0}^{\infty} r^n,$$

in which the ratio of any term to the preceding term is a constant, r, is called a **geometric series.** Let us investigate the convergence and non-convergence of this series for real values of r.

By Algebra, we have for the sum of n terms,

$$S_n = \frac{1 - r^n}{1 - r} = \frac{1}{1 - r} - \frac{r^n}{1 - r}.$$

Now if $|r| < 1$, by Art. **43,** $Lr^n = 0$, and we have $LS_n = \frac{1}{1 - r} = S$, and the series is convergent with sum $= \frac{1}{1 - r}$. If $|r| > 1$, then by Art. **43,** $|r|^n \to \infty$; and if $r > 1$, $Lr^n = +\infty$, and $LS_n = +\infty$, so that the series is properly divergent; if $r < -1$, r^n evidently oscillates between positive and negative values and numerically $\to \infty$, so that then the series is improperly divergent. If $r = +1$, the series is $1 + 1 + 1 + 1 + \dots$, which is properly divergent. If $r = -1$, we have the series $1 - 1 + 1 - 1 + \dots$, which is oscillating. Hence, we may summarize:

The geometric series $\sum_{0}^{\infty} r^n$ *is:*

Convergent if $|r| < 1$, *and its sum is then* $\frac{1}{1 - r}$;

Properly divergent if $r > 1$ *or* $r = 1$;
Improperly divergent if $r < -1$.
Oscillating if $r = -1$.

76. The Harmonic Series.—*The series*

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ .\ .\ .\ +\frac{1}{n}+\ .\ .\ .\ \equiv\sum_{1}^{\infty}\frac{1}{n}$$

is called the **harmonic series.** Let us group its terms as follows:

$1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + (1/9 + 1/10 + \ .\ .\ .\ + 1/16) + \ .\ .\ .$

The sum of the terms in each parenthesis is $> 1/2$, so that by taking enough of these groups we can evidently make the sum of these terms as large as we please, and hence the sum of n terms $S_n \to \infty$ as $n \to \infty$. Therefore, the harmonic series is properly divergent. The divergence of this series also follows from Art. **61** (*a*).

The divergence of this series is comparatively slow; the sum to a million terms is < 21.

77. The Hyperharmonic Series.—*The series*

$$1+\frac{1}{2^k}+\frac{1}{3^k}+\ .\ .\ .\ +\frac{1}{n^k}+\ .\ .\ .\ \equiv\sum_{1}^{\infty}\frac{1}{n^k}$$

is called the **hyperharmonic series**;[1] it includes the harmonic series as a special case, for $k = 1$.

Group its terms as follows: x^{k-1}

$$(a)\quad 1+\left(\frac{1}{2^k}+\frac{1}{3^k}\right)+\left(\frac{1}{4^k}+\frac{1}{5^k}+\frac{1}{6^k}+\frac{1}{7^k}\right)+\left(\frac{1}{8^k}+\ .\ .\ .\ +\frac{1}{15^k}\right)+\ .\ .\ .$$

Now if $k > 1$, we have

$$\frac{1}{2^k}+\frac{1}{3^k}<\frac{1}{2^k}+\frac{1}{2^k}=\frac{2}{2^k}=\frac{1}{2^{k-1}},$$

$$\frac{1}{4^k}+\frac{1}{5^k}+\frac{1}{6^k}+\frac{1}{7^k}$$

---

[1] The name "hyperharmonic" is due to PIERPONT, *cf.* his "Functions of a Complex Variable," art. **14.**

$$< \frac{1}{4^k} + \frac{1}{4^k} + \frac{1}{4^k} + \frac{1}{4^k} = \frac{4}{4^k} = \frac{1}{(2^{k-1})^2},$$

$$\frac{1}{8^k} + \frac{1}{9^k} + \dots + \frac{1}{15^k}$$

$$< \frac{1}{8^k} + \frac{1}{8^k} + \dots + \frac{1}{8^k} = \frac{8}{8^k} = \frac{1}{(2^{k-1})^3},$$

. . . .

Thus each group of terms in (a) is $<$ the corresponding term of the series

$$1 + \frac{1}{2^{k-1}} + \frac{1}{(2^{k-1})^2} + \frac{1}{(2^{k-1})^3} + \dots,$$

which is a geometric series with ratio $r = \frac{1}{2^{k-1}} < 1$, hence this geometric series is convergent with sum g say, then if S_n be the sum of n terms of the series (a), we have $S_n < g$ for every n. Now the sequence (S_n) is increasing since the terms of (a) are all positive, and (S_n) is bounded since $S_n < g$, hence by Art. **28** the sequence (S_n) is convergent, and consequently the hyperharmonic series is convergent for $k > 1$.

If $k = 1$, we have the harmonic series which has already been shown to be properly divergent.

If $k < 1$, each term $\frac{1}{n^k}$ of the hyperharmonic series is $>$ the corresponding term $\frac{1}{n}$ of the harmonic series, so that if S_n is the sum of n terms of $\sum \frac{1}{n^k}$ and H_n is the sum of n terms of $\sum \frac{1}{n}$, we have $S_n > H_n$. Since $H_n \to \infty$ as $n \to \infty$, we must have $S_n \to \infty$ (Art. **37**), and the series $\sum \frac{1}{n^k}$ is properly divergent in this case.

Hence, *the hyperharmonic series* $\sum \frac{1}{n^k}$ *is convergent when* $k > 1$, *and properly divergent when* $k \leqq 1$.

78. General Principle of Convergence for Series.—To use the definition of convergence as a test to determine whether a given series is convergent requires a knowledge of the sum or the possession of a formula for the sum of n terms, and hence it is not well adapted for practical use. The general principle of convergence for sequences applied to series gives a much more practicable test. It may be stated:

Theorem.—*A general necessary and sufficient condition for the convergence of a series* $\sum u_n$ *is that, for every* $\epsilon > 0$, *however small, an index* ν *can be found such that for every* $n > \nu$ *and for every* p, *we have*

$$(a) \qquad |U_{n+p} - U_n| < \epsilon, \text{ or } |U_{n,p}| < \epsilon.$$

This inequality may also be written

$$(b) \qquad |u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \epsilon.$$

This theorem may be stated in another form:

A necessary and sufficient condition for the convergence of the series $\sum u_n$ *is that* $\underset{n}{L} U_{n,p} = 0$ *for every* p.

The condition can often be applied indirectly to show that a series is not convergent. For if we can show that $\underset{n}{L} U_{n,p} \neq 0$ for some p, we know then that the series cannot be convergent. For example, in the harmonic series, the $U_{n,n}$ is

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{n}{2n} = \frac{1}{2},$$

so that $\underset{n}{L} U_{n,n} \neq 0$ for $p = n$, and hence the series cannot be convergent, and since its terms are all positive, it must be properly divergent, as shown before.

79. Necessary Condition for Convergence.—If in Art. 78 we put $p = 1$, we have $U_{n,1} = u_{n+1}$, so that a necessary condition for convergence is $Lu_{n+1} = 0$, or what amounts to the same thing, $Lu_n = 0$; that is, as n increases indefinitely, the general term of the series must approach zero.

Theorem.—*A necessary condition for convergence of a series* $\sum u_n$ is

nec.

$$Lu_n = 0.$$

This condition is **not sufficient** for convergence, as is shown by the fact that for the divergent harmonic series we have $Lu_n = L\frac{1}{n} = 0$.

If we find that $Lu_n \neq 0$, we know at once that the series is non-convergent, and this may often be used as a test of non-convergence. For example, in the series

$$\tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} + \tfrac{4}{5} + \cdots,$$

we have $Lu_n = 1$, hence the series is properly divergent.

80. General Test for Positive Series.—If the terms of a series $\sum a_n$ are all positive, then the sequence (A_n) is increasing, and by Arts. **28, 29** we see that the series must be either convergent or properly divergent, and if (A_n) is bounded, then the series must be convergent.

Theorem.—*A positive series $\sum a_n$ is convergent if A_n is less than some fixed number for every n. The series is properly divergent if for any positive number G, however large, we have $A_n > G$ for every $n >$ some index ν.*

81. The Alternating Series Test.—A series whose terms are alternately positive and negative is called an **alternating series.**

Theorem.—*An alternating series is convergent if each term is numerically less than the preceding, and the limit of the n^{th} term is zero.*

Let the series be represented by

$$a_1 - a_2 + a_3 - a_4 + a_5 - \cdots,$$

where the a's are all positive and form a decreasing sequence whose limit is 0. Then

$$|A_{n,p}| = |a_{n+1} - a_{n+2} + a_{n+3} - \cdot \cdot \cdot + (-1)^{p-1}a_{n+p}|.$$

We may write

$$(a) \qquad a_{n+1} - a_{n+2} + a_{n+3} - \cdot \cdot \cdot + (-1)^{p-1}a_{n+p}$$

in the form

$$(a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \cdot \cdot \cdot + (a_{n+p-1} - a_{n+p})$$

if p is even, or

$$(a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \cdot \cdot \cdot + (a_{n+p-2} - a_{n+p-1}) + a_{n+p}$$

if p is odd.

Since the sequence (a_n) is decreasing, each of the parentheses above is positive, so that the expression (a) is positive in either case, and we may write

$$|A_{n,p}| = a_{n+1} - a_{n+2} + a_{n+3} - \cdot \cdot \cdot + (-1)^{p-1}a_{n+p}.$$

But (a) may also be written

$$a_{n+1} - (a_{n+2} - a_{n+3}) - \cdot \cdot \cdot - (a_{n+p-2} - a_{n+p-1}) - a_{n+p}$$

for p even, or

$$a_{n+1} - (a_{n+2} - a_{n+3}) - \cdot \cdot \cdot - (a_{n+p-1} - a_{n+p})$$

for p odd; so that for p odd or even,

$$(b) \qquad |A_{n,p}| < a_{n+1}.$$

But since by hypothesis, we have $La_{n+1} = 0$, or

$$\epsilon > 0,\ \nu,\ a_{n+1} < \epsilon,\ n > \nu,$$

we get by (b),

$$\epsilon > 0,\ \nu,\ |A_{n,p}| < \epsilon,\ n > \nu,\ p = 1, 2, 3, \ldots,$$

and by the general principle of convergence, the series is convergent.

This test is a sufficient but not a necessary condition for convergence.

Examples.—(1) The series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \ . \ . \ .$$

satisfies the conditions of this test and is therefore convergent.

(2) The series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ . \ . \ .$$

is convergent by this test.

(3) The series

$$A = \frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}-1} - \frac{1}{\sqrt{3}+1} + \ . \ . \ .$$

has its terms alternately positive and negative, and $a_n \rightarrow 0$ but it is divergent. For

$$\begin{aligned} A_{2m} &= \sum_{i=2}^{m}\left(\frac{1}{\sqrt{i}-1} - \frac{1}{\sqrt{i}+1}\right) = \sum_{i=2}^{m}\frac{2}{i-1} \\ &= 2\left(1 + \frac{1}{2} + \frac{1}{3} + \ . \ . \ . \ + \frac{1}{m-1}\right) = 2H_{m-1}, \end{aligned}$$

which $\rightarrow \infty$ as $m \rightarrow \infty$. If the series were convergent, $\underset{n}{L}A_n$ would be equal to $\underset{m}{L}A_{2m}$, hence the series must be divergent. In this series, the condition that the terms decrease steadily is not satisfied.

82. The Limit of Error in the Alternating Series.—It is evident from the proof of Art. **81** that the remainder after n terms, $\overline{A}_n$, is less than the term a_{n+1}, and hence if we take the sum of n terms of the series as an approximation to the sum, the error made is less than the next term, and hence approaches zero, as we take more and more terms; and if a certain definite number of terms are added, the error can be determined definitely. In other words, we may stop adding terms as soon as we come to a term which is numerically less than the proposed limit of error.

For example, let it be required to compute the value of the series

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \quad . \quad . \quad . \quad ,$$

correct to three decimal places. We have

$$\begin{array}{llll}
\dfrac{1}{2!} = .5000 & \qquad\qquad & \dfrac{1}{3!} = .1667 \\
\dfrac{1}{4!} = .0417 & & \dfrac{1}{5!} = .0083 \\
\dfrac{1}{6!} = .0014 & & \dfrac{1}{7!} = \dfrac{.0002}{.1752} \\
\dfrac{1}{8!} = \dfrac{.0000}{.5431} & &
\end{array}$$

$$.5431 - .1752 = .3679.$$

The term $\frac{1}{8!}$ being less than the required limit of error, we stop with that term and can be certain that all the terms after that cannot affect the result in the first three decimal places, so that to three places the value of the sum of the series is .368. The fourth decimal place is retained throughout the computation to insure accuracy in the third place in the final result.

83. Theorem.—*A series* $\sum u_n$ *of real or complex terms is convergent if the series of the absolute values of its terms is convergent.*

For, if the series $\sum |u_n|$ is convergent, then by Art. **78** we have

$$\epsilon > 0, \ \nu, \ \{ |u_{n+1}| + |u_{n+2}| + \ . \ . \ . \ + |u_{n+p}| \} < \epsilon,$$

for $n > \nu$, and for every p. But if $U_{n,p}$ is the partial remainder of the series $\sum u_n$, we have

$$|U_{n+p}| = |u_{n+1} + \ldots + u_{n+p}| \leq |u_{n+1}| + \ldots + |u_{n+p}| < \epsilon$$

for $n > \nu$, $p = 1, 2, 3, \ldots$ Hence, $\sum u_n$ is convergent (Art. **78**).

Example.—The series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$ is convergent since the series of absolute values $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is convergent. It is also seen to be convergent by the alternating series test, and by the geometric series test.

84. Absolute Convergence.—*A series is said to be* **absolutely convergent** *if the series of absolute values of its terms is convergent* (Art. **83**). *Series which do not converge absolutely are called* **simply convergent.**

Examples.—The series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$ is absolutely convergent since $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is convergent; but the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is only simply convergent since $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent.

85. Composition of Positive and Negative Term Series.—Let $\sum u_n$ be any given real series; if it contains only a finite number of positive (or of negative) terms, it is essentially a negative (or positive) series, and hence must be either convergent or properly divergent. Suppose then that $\sum u_n$ has an infinite number of positive terms and an infinite number of negative terms. Let the positive terms, in the order in which they occur, be $a_1, a_2, a_3, \ldots, a_n, \ldots$, and the negative terms, in order, $-b_1, -b_2, -b_3, \ldots, -b_n, \ldots$ Now consider the two series $\sum a_n$, $\sum b_n$, each of which is a positive series. If U_n contains m positive terms and m' negative terms, we have

$$(a) \qquad U_n = A_m - B_{m'}.$$

1° If both positive series $\sum a_n$, $\sum b_n$ converge, then $\underset{m}{L} A_m$, $\underset{m'}{L} B_{m'}$ both exist with finite values, and therefore $\underset{n}{L} U_n$ exists $= A - B$, and therefore $\sum u_n$ is convergent. In this case, $\sum u_n$ is absolutely convergent, for if σ_n be the sum of n terms of $\sum |u_n|$, we have $\sigma_n = A_m + B_{m'}$, and since A_m, $B_{m'}$ both approach finite limits, σ_n must approach a finite limit as $n \to \infty$, and hence $\sum |u_n|$ is convergent, and therefore $\sum u_n$ is absolutely convergent.

2° If one of the series $\sum a_n$, $\sum b_n$ is convergent and the other divergent, then the series $\sum u_n$ is evidently divergent.

3° If both series $\sum a_n$, $\sum b_n$ are divergent, then it may happen that $\sum u_n$ is convergent; for $A_m \to \infty$ as $m \to \infty$, and $B_{m'} \to \infty$ as $m' \to \infty$, but $A_m - B_{m'} = U_n$ may have a limit (for U_n takes the indeterminate form $\infty - \infty$). But in this case, $\sigma_n = (A_m + B_{m'}) \to \infty$, and hence $\sum |u_n$ is divergent, and therefore $\sum u_n$ is only simply convergent, if convergent at all.

From this, we get the

Theorem.—*If* $\sum u_n$ *is made up of the positive terms* (a_n) *and the negative terms* $(-b_n)$, *and if* $\sum a_n$ *and* $\sum b_n$ *are both convergent, then* $\sum u_n$ *is absolutely convergent; if both* $\sum a_n$, $\sum b_n$ *are divergent, then* $\sum u_n$ *is at most simply convergent (if convergent at all); and conversely, if* $\sum u_n$ *is simply convergent, then* $\sum a_n$, $\sum b_n$ *must be divergent; and if* $\sum u_n$ *is absolutely convergent, then* $\sum a_n$, $\sum b_n$ *are both convergent. If one of the* *series* $\sum a_n$, $\sum b_n$ *is convergent and one divergent, then* $\sum u_n$ *is divergent.*

86. Telescopic Series.—This name is given[1] to series of the form

$$(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \ . \ . \ . \ .$$

We have

$$\begin{aligned} A_n &= (a_1 - a_2) + (a_2 - a_3) + \ . \ . \ . \ + (a_{n-1} - a_n) + (a_n - a_{n+1}) \\ (a) \qquad &= a_1 - a_{n+1}. \end{aligned}$$

From this the reason for the name is evident. We see also that a telescopic series is convergent when and only when La_n exists.

The use of telescopic series comes from the fact that any series may be written in the telescopic form. For if $\sum a_n$ be a given series, then $a_n = A_n - A_{n-1}$, $a_1 = A_1$, and

$$\sum a_n = (A_2 - A_1) + (A_3 - A_2) + (A_4 - A_3) + \ . \ . \ .$$

Examples.—(1) The series

$$A = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ . \ . \ . \ = \sum_1^\infty \frac{1}{n(n+1)}$$

can be written in the telescopic form, giving

$$A_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ . \ . \ . \ \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1},$$

hence $LA_n = 1$, and the series is convergent with sum $= 1$.

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[1] By PIERPONT, *cf.* his "Theory of Functions of Real Variables," vol. II, p. 85.

(2) For the series

$$B = \frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \ . \ . \ . \ = \sum_{1}^{\infty} \frac{1}{n(n+2)},$$

we have

$$B_n = \sum^{n} \frac{1}{m(m+2)} = \frac{1}{2} \sum_{1}^{n} \left(\frac{1}{m} - \frac{1}{m+2}\right) =$$

$$\frac{1}{2}\left\{\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \ . \ . \ . \ + \left(\frac{1}{n} - \frac{1}{n+2}\right)\right\}$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right),$$

which $\rightarrow \frac{3}{4}$, hence the series is convergent with sum $= \frac{3}{4}$.

87. Addition and Subtraction of Series.—Given two series $\sum u_n$ and $\sum v_n$, to add or subtract these series is to form the new series $\sum w_n$, where $w_n = u_n \pm v_n$.

Theorem.—*If* $\sum u_n$ *and* $\sum v_n$ *are convergent and have the sums* U *and* V, *then the series* $\sum (u_n \pm v_n)$ *is convergent and has for its sum* $U \pm V$.

For

$$\sum_{1}^{n} w_i = \sum_{1}^{n} \left(u_i \pm v_i \right) = \sum_{1}^{n} u_i \pm \sum_{1}^{n} v_i,$$

and we have only to pass to the limit as $n \rightarrow \infty$.

This result can be extended to any *finite* number of series, and we may write

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{k} u_i^{(j)}\right) = \sum_{j=1}^{k} \left(\sum_{i=1}^{\infty} u_i^{(j)}\right).$$

88. Two-way Series.—Series sometimes arise which extend to ∞ both ways; they are of the type

$$. \ . \ . \ a_{-3} + a_{-2} + a_{-1} + a_0 + a_1 + a_2 + a_3 + \ . \ . \ . \ ,$$

which may be written

$$\sum_{-\infty}^{+\infty} a_n.$$

Such series are called **two-way series.**

This series is convergent if $\underset{r,s}{L} \sum_{n=-r}^{s} a_n$ exists with a finite value, where the limit is taken when r and s both become infinite independently of each other. If this limit does not exist, the series is non-convergent.

Sometimes the value $n = 0$ is to be excluded; this is ordinarily indicated by an accent thus: $\sum_{-\infty}^{\infty}{}' a_n$.

When n ranges over the set of values . . . , -3, -2, -1, 0, 1, 2, 3, . . . , if m is any integer, $i = n + m$ will range over the same set except that i will be m units ahead or behind n according as $m \gtrless 0$. Hence we may write

$$\sum_{n=-\infty}^{+\infty} a_n = \sum_{n=-\infty}^{+\infty} a_{n+m},$$

and similarly

$$\sum_{-\infty}^{+\infty} a_n = \sum_{-\infty}^{+\infty} a_{-n}$$

Examples.—The series $\sum_{-\infty}^{+\infty} e^{nx+pn^2}$,

and

$$\frac{1}{x} + \sum_{-\infty}^{+\infty}{}' \left(\frac{1}{x+n} - \frac{1}{n} \right)$$

are examples of two-way series.

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CHAPTER VIII

SIMPLE TESTS OF CONVERGENCE FOR POSITIVE SERIES

89. Criteria of Convergence.—There exists no practicable universal test for determining whether any given series is convergent or not, but there are a great number of tests, each applying to a limited class of series. Most of the criteria are expressed as applicable to positive series; if we wish to test a series in general, we usually take its absolute value series and apply to this the positive series criteria and thus test the absolute convergence of the series.

90. Positive Series.—Since positive series must be either convergent or properly divergent, when we use the term "divergent" in connection with positive series, we shall always mean properly divergent. In testing a series for convergence, we may omit any finite number of terms at the beginning, for oftentimes a series has certain irregularities in its first few terms which render the application of the tests difficult.

As far as practicable we shall use the first letters of the alphabet for positive series, as $\sum a_n$, $\sum b_n$, etc., and the later letters for general series with positive and negative, or complex terms, as $\sum u_n$, $\sum v_n$, etc. We shall use $\sum c_n$ to represent a positive series known to be convergent, and $\sum d_n$ for a positive series known to be divergent.

91. Comparison Tests.—Among the simpler tests of convergence of positive series are the comparison tests, in which we compare the terms of a given series with the corresponding terms of a series whose character, as to convergence or divergence, is known. For this purpose, it is necessary to have some standard series whose character is known, to use as comparison series. We have already investigated the following series, which are useful as comparison series: (1) the geometric series $\sum r^n$, which is convergent for $|r| < 1$, divergent for $|r| \geqq 1$; (2) the harmonic series $\sum \frac{1}{n}$, which is divergent; (3) the hyperharmonic series $\sum \frac{1}{n^k}$, which is convergent for $k > 1$, divergent for $k \leqq 1$; and (4) the series $\sum \frac{1}{n(n+1)}$, which is convergent (Art. **86**). These series are then available already as standard comparison series, and others will be obtained later.

92. Theorem.—*If $a_n \leqq c_n$ for every n (or at least for every n after a certain value),*[1] *then $\sum a_n$ is convergent.*

For $A_n \leqq C_n < C$, and by Art. **80**, $\sum a_n$ must be convergent.

Examples.—(1) $\sum_1^\infty \frac{1}{n^2}$ is convergent, since $\frac{1}{n^2} < \frac{1}{n(n-1)}$, and $\sum \frac{1}{n(n-1)}$ is known to be convergent. We have thus a second proof of the convergence of the hyperharmonic series for $k = 2$.

(2) $\sum_1^\infty \frac{1}{n!}$ is convergent, since $\frac{1}{n!} < \frac{1}{2^{n-1}}$, and $\sum \frac{1}{2^{n-1}}$ is a geometric series which is convergent.

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[1] If the relation is not true for every n at the beginning of the series, we may omit these first few terms for which it is not true, and thus make the relation true for *every* n in the new series, and thus avoid the necessity for the statement in parentheses. We shall suppose this done hereafter.

93. Theorem.—*If* $a_n \geqq d_n$ *for every* n, *then* $\sum a_n$ *is divergent.*

For,

$$G > 0,\ \nu,\ D_n > G,\ n > \nu,$$

and $A_n \geqq D$, hence $A_n > G$, and $\sum a_n$ is divergent.

Examples.—(1) $\sum \frac{1}{\sqrt{n}}$ is divergent, since $\frac{1}{\sqrt{n}} > \frac{1}{n}$, and $\sum \frac{1}{n}$ is divergent.

(2) $\sum \frac{1}{\sqrt{n(n+1)}}$ is divergent, since $\frac{1}{\sqrt{n(n+1)}} > \frac{1}{n+1}$.

94. Theorem.—*If* $\frac{a_n}{c_n} < k$ *(a finite constant) for every* n, *then* $\sum a_n$ *is convergent.*

For $\frac{a_n}{c_n} < k$ gives $a_n < kc_n$, and we may apply Arts. **74, 92.**

Example.—$\sum \frac{1}{n(n+2)}$ is convergent since $\frac{1}{n(n+2)} \div \frac{1}{n^2} = \frac{n}{n+2} < 1$, and $\sum \frac{1}{n^2}$ is convergent (*cf.* Art. **86**, ex. 2).

95. Theorem.—*If* $\frac{a_n}{d_n} > k$ *(a finite constant) for every* n, *then* $\sum a_n$ *is divergent.*

For $a_n > kd_n$, and we may apply Arts. **74, 93.**

Example.—$\sum \frac{n+3}{n(n+2)}$ is divergent since

$$\frac{n+3}{n(n+2)} \div \frac{1}{n} = \frac{n+3}{n+2} > 1.$$

96. Theorem.—*If* $L \frac{a_n}{c_n}$ *exists (a finite constant)* $\neq 0$, then $\sum a_n$ *is convergent.*

For, if $L\dfrac{a_n}{c_n} = l$, we may write $\dfrac{a_n}{c_n} = l + \delta_n$, where $\delta_n \to 0$, then a constant δ can be found such that $|\delta_n| < \delta$ for every n, and therefore $\dfrac{a_n}{c_n} < l + \delta$, and we may apply Art. **92.**

Example.—$\sum \dfrac{2n+1}{n^3+n}$ is convergent since

$$L\left(\frac{2n+1}{n^3+n} \div \frac{1}{n^2}\right) = L\,\frac{2+\dfrac{1}{n}}{1+\dfrac{1}{n^2}} = 2.$$

97. Theorem.—*If* $L\dfrac{a_n}{d_n}$ *exists (a finite constant)* $\neq 0$, *then* $\sum a_n$ *is divergent.*

For, if $L\dfrac{a_n}{d_n} = l'$, we have $\dfrac{a_n}{d_n} = l' + \delta'_n$, where $\delta'_n \to 0$ and $\delta' > 0$ can be found such that every $\delta'_n > -\delta'$, and then $\dfrac{a_n}{d_n} > l' - \delta'$, which reduces the problem to Art. **95.**

Example.—$\sum \dfrac{2n-1}{(n+1)(n+2)}$ is divergent since

$$L\left(\frac{2n-1}{(n+1)(n+2)} \div \frac{1}{n}\right) = L\,\frac{(2n-1)n}{(n+1)(n+2)} = 2.$$

98. Theorem.—*If* $a_n = \dfrac{f(n)}{g(n)}$, *where* $f(n)$ *and* $g(n)$ *are rational integral functions of* n, *and the degree of* $g(n)$ *exceeds that of* $f(n)$ *by more than* 1, *then* $\sum a_n$ *is convergent, otherwise divergent.*

For, we may write

$$\begin{aligned} a_n &= \frac{M_0 n^p + M_1 n^{p-1} + \ldots + M_p}{N_0 n^{p+k} + N_1 n^{p+k-1} + \ldots + N_{p+k}} \\ &= \frac{1}{n^k}\cdot\frac{M_0 n^p + M_1 n^{p-1} + \ldots + M_p}{N_0 n^p + N_1 n^{p-1} + \ldots + N_{p+k}\cdot n^{-k}}, \end{aligned}$$

and

$$L\left(a_n \div \frac{1}{n^k}\right) = L\,\frac{M_0 n^p + \quad . \quad . \quad . \quad + M_p}{N_0 n^p + \quad . \quad . \quad . \quad + N_{p+k} \cdot n^{-k}} = \frac{M_0}{N_0} \text{ (const.)},$$

hence by Art. **96**, the series is convergent for $k > 1$.

Example.—$\sum \frac{2n+1}{(n-1)n(n+1)}$ satisfies the conditions of the theorem, and so is convergent.

99. Theorem.—*If* $\frac{a_{n+1}}{a_n} \leqq \frac{c_{n+1}}{c_n}$ *for every* n, *then* $\sum a_n$ *is convergent.*

For, from $\frac{a_{n+1}}{a_n} \leqq \frac{c_{n+1}}{c_n}$, we get

$$\frac{a_{n+1}}{c_{n+1}} \leqq \frac{a_n}{c_n} \leqq \frac{a_{n-1}}{c_{n-1}} \leqq \quad . \quad . \quad . \quad \leqq \frac{a_1}{c_1},$$

and we may apply Art. **94**.

Example.—$\sum_1^\infty \frac{1}{n!}$ is convergent, since in this series

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)!} \div \frac{1}{n!} = \frac{1}{n+1},$$

and in the convergent series $\sum_1^\infty \frac{1}{2^{n-1}}$, we have

$$\frac{c_{n+1}}{c_n} = \frac{1}{2^n} \div \frac{1}{2^{n-1}} = \frac{1}{2}, \text{ and } \frac{1}{n+1} < \frac{1}{2}.$$

100. Theorem.—*If* $\frac{a_{n+1}}{a_n} \geqq \frac{d_{n+1}}{d_n}$ *for every* n, *then* $\sum a_n$ *is divergent.*

For

$$\frac{a_{n+1}}{d_{n+1}} \geqq \frac{a_n}{d_n} \geqq \quad . \quad . \quad . \quad \geqq \frac{a_1}{d_1},$$

and Art. **90** applies.

Example.—The series

$$A = \frac{1}{2} + \frac{1\cdot 3}{2\cdot 4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \dots + \frac{1\cdot 3\cdot 5 \dots 2n-1}{2\cdot 4\cdot 6 \dots 2n} + \dots$$

has the ratio $\frac{a_{n+1}}{a_n} = \frac{2n+1}{2n+2}$, and $\sum \frac{1}{n} = D$ has the ratio $\frac{d_{n+1}}{d_n} = \frac{n}{n+1}$, and $\frac{2n+1}{2n+2} > \frac{n}{n+1}$, hence A is divergent.

101. D'Alembert's Ratio Test.[1] **Theorem.**—*If* $\frac{a_{n+1}}{a_n} < r < 1$ *for every* n, *then* $\sum a_n$ *is convergent, and if* $\frac{a_{n+1}}{a_n} \geqq 1$, *then* $\sum a_n$ *is divergent.*

For, if $\frac{a_{n+1}}{a_n} < r$, we can apply Art. **99,** taking the geometric series $\sum r^n$ for the comparison series $\sum c_n$, which is convergent for $r < 1$. If $\frac{a_{n+1}}{a_n} \geqq 1$, we have $a_{n+1} \geqq a_n$, and a_n cannot $\to 0$, hence (Art. **79**) $\sum a_n$ must be divergent.

Example.—The series

$$A = \frac{3}{5} + \frac{3\cdot 5}{5\cdot 10} + \frac{3\cdot 5\cdot 7}{5\cdot 10\cdot 15} + \dots + \frac{3\cdot 5\cdot 7 \dots 2n+1}{5\cdot 10\cdot 15 \dots 5n} + \dots$$

is convergent since

$$\frac{a_{n+1}}{a_n} = \frac{2n+1}{5n} = \frac{2+\frac{1}{n}}{5} < \frac{3}{5} < 1.$$

In applying this test, we must not only show that $\frac{a_{n+1}}{a_n}$ is < 1 for every value of n (at least after a certain term), but we must show that the ratio $\frac{a_{n+1}}{a_n}$ is $<$ some number r which is itself a fixed number < 1. To illustrate the need for this, consider the series

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[1] This test is often called *Cauchy's ratio test.*

$$1 + 2x + 3x^2 + 4x^3 + \ . \ . \ . \ + nx^{n-1} + \ . \ . \ .$$

The ratio $\frac{a_n}{a_{n-1}} = \frac{nx}{n-1}$, and this ratio will not be < 1 until $\frac{nx}{n-1} < 1$ or $n > \frac{1}{1-x}$. Here we have a convergent series in which the terms increase up to a certain point, and then begin to decrease. The value of n which makes the ratio < 1 depends on the value of x; thus, for $x = .99$, $\frac{1}{1-x} = 100$, and the terms do not begin to decrease until after the 100th term, and the ratio $\frac{a_n}{a_{n-1}}$ does not become < 1 until after the 100th term.

The ratio $\frac{a_{n+1}}{a_n}$ may be < 1 for divergent series, as, for example, in the harmonic series $\sum \frac{1}{n}$, where the ratio is $1 - \frac{1}{n}$, but in such cases it approaches a limit 1, and does not satisfy the condition $\frac{a_{n+1}}{a_n} < r < 1$.

The following form of the test is often found more convenient in practice.

Theorem.—*If* $L \frac{a_{n+1}}{a_n}$ *exists* $= \rho$, *then* $\sum a_n$ *is convergent when* $\rho < 1$, *divergent when* $\rho > 1$, *and no test is given when* $\rho = 1$, *except when* $\frac{a_{n+1}}{a_n} > 1$ *for every* n, *in which case the series is divergent.*

Suppose first that $\rho < 1$, then if r be any number between ρ and 1, we can find ν such that $\left|\frac{a_{n+1}}{a_n} - \rho\right| < r - \rho$ for

$n > \nu$, from the definition of limit, hence we get $\frac{a_{n+1}}{a_n} < r$, $n > \nu$. Now from $\frac{a_{n+1}}{a_n} < r < 1$, we see from the first form of the test that $\sum a_n$ is convergent.

If $\rho > 1$, then $\frac{a_{n+1}}{a_n} = \rho + \delta_n$, where $\delta_n \to 0$, and for n sufficiently large, say $n > \nu$, we can make $\rho + \delta_n > 1$, so that $\frac{a_{n+1}}{a_n} > 1$, and we can apply the preceding form of the test, which shows that $\sum a_n$ is divergent.

If $\rho = 1$ and $\frac{a_{n+1}}{a_n} > 1$ for every n, we have $a_{n+1} > a_n$, and a_n cannot $\to 0$, so that $\sum a_n$ must be divergent.

If $\rho = 1$ and $\frac{a_{n+1}}{a_n} \ngtr 1$ for every n, then the ratio test fails and some other test must be used. For ρ may $= 1$, for both convergent and divergent series, as, for example, in the convergent series $\sum \frac{1}{n^2}$, in which

$$L\frac{a_{n+1}}{a_n} = L\frac{1}{\left(1 + \frac{1}{n}\right)^2} = 1,$$

and also for the divergent series $\sum \frac{1}{n}$, in which

$$L\frac{a_{n+1}}{a_n} = L\frac{1}{\left(1 + \frac{1}{n}\right)} = 1.$$

Examples.—(1) The series

$$A = 1 + \frac{1\cdot 3}{1\cdot 4} + \frac{1\cdot 3\cdot 5}{1\cdot 4\cdot 7} + \ . \ . \ .$$

is convergent, since

$$L\frac{a_{n+1}}{a_n} = L\frac{2n+1}{3n+1} = \frac{2}{3} < 1.$$

(2) The series $B = \sum \frac{1}{n(n+3)}$ has

$$L \frac{b_{n+1}}{b_n} = L \frac{n(n+3)}{(n+1)(n+4)} = 1,$$

so the ratio test fails, but by comparing with $\sum \frac{1}{n^2}$, we have $\frac{1}{n(n+3)} < \frac{1}{n^2}$, hence B is convergent.

The ratio test is perhaps more widely used than any other test.

By using the conception of extreme limits, we may state the ratio test in a more general form, as follows:

Theorem.—*If* $\underline{L} \frac{a_n}{a_{n+1}} > 1$, *then* $\sum a_n$ *is convergent, and if* $\overline{L} \frac{a_n}{a_{n+1}} < 1$, *then* $\sum a_n$ *is divergent.*

For, if $\underline{L} \frac{a_n}{a_{n+1}} > 1$, we can find a constant $t > 1$ such that for every n, $\frac{a_n}{a_{n+1}} > t > 1$ by definition of lower limit (Art. **18**), or

$$\frac{a_{n+1}}{a_n} < \frac{1}{t} < 1,$$

hence by the first form of the ratio test, $\sum a_n$ is convergent.

If $\overline{L} \frac{a_n}{a_{n+1}} < 1$, a constant t' can be found such that for every n, $\frac{a_n}{a_{n+1}} < t' < 1$, or

$$\frac{a_{n+1}}{a_n} > \frac{1}{t'} > 1,$$

so that $\sum a_n$ is divergent.

In case $L \frac{a_n}{a_{n+1}} = 1$ and $\frac{a_n}{a_{n+1}} \geq 1$, the ratio test fails, and we then have recourse to the tests of Arts. **108**, etc.

102. Limit of Error.—If for a given series $\sum a_n$ we have $\frac{a_{n+1}}{a_n} < r < 1$, and we compute the sum of the first m terms as an approximation to the sum of the series, the error made in this approximation will be

$$a_{m+1} + a_{m+2} + a_{m+3} + \ . \ . \ . \ ,$$

but since $\frac{a_{n+1}}{a_n} < r$, we have

$$a_{m+1} < ra_m,\ a_{m+2} < r^2a_m,\ a_{m+3} < r^3a_m,\ . \ . \ . \ ,$$

so that the error is less than the sum of the convergent geometric series

$$\begin{aligned} ra_m + r^2a_m + r^3a_m + \ . \ . \ . \ &= a_m \cdot r\,(1 + r + r^2 + \ . \ . \ . \ ) \\ &= a_m \cdot \frac{r}{1-r}. \end{aligned}$$

Hence, the error made in stopping with the term a_m is $< a_m \cdot \frac{r}{1-r}$.

Example.—In the series

$$1 + \frac{1\cdot3}{1\cdot7} + \frac{1\cdot3\cdot5}{1\cdot7\cdot10} + \frac{1\cdot3\cdot5\cdot7}{1\cdot7\cdot10\cdot13} + \frac{1\cdot3\cdot5\cdot7\cdot9}{1\cdot7\cdot10\cdot13\cdot16} + \ . \ . \ . \ ,$$

we have

$$\frac{a_{n+1}}{a_n} = \frac{2n-1}{3n+1} = \frac{2-\frac{1}{n}}{3+\frac{1}{n}} < \frac{2}{3},$$

and we may take $r = 2/3 < 1$, and the error made in taking five terms for an approximation to the sum would be less than

$$\frac{1\cdot3\cdot5\cdot7\cdot9}{1\cdot7\cdot10\cdot13\cdot16} \cdot \frac{2/3}{1-2/3} = \frac{1\cdot3\cdot5\cdot7\cdot9}{1\cdot7\cdot10\cdot13\cdot16} \cdot 2 = \frac{27}{208} = .129+$$

103. Cauchy's Radical Test. **Theorem.**—*If for every* n, $\sqrt[n]{a_n} < r < 1$, *then* $\sum a_n$ *is convergent, and if* $\sqrt[n]{a_n} \geqq 1$, *then* $\sum a_n$ *is divergent.*

For, if $\sqrt[n]{a_n} < r$, we have $a_n < r^n$, and each term of $\sum a_n$ is $\leqq$ the corresponding term of the convergent geometric series $\sum r^n$, and hence (Art. 92) $\sum a_n$ is convergent.

If $\sqrt[n]{a_n} \geqq 1$, we have $a_n \geqq 1$, and a_n cannot $\rightarrow 0$, hence $\sum a_n$ must be divergent.

The following form of the test is often more convenient.

Theorem.—*If* $L\sqrt[n]{a_n}$ *exists* $= \lambda$, *then* $\sum a_n$ *is convergent when* $\lambda < 1$, *and divergent when* $\lambda > 1$; *if* $\lambda = 1$, *there is no test, unless* $\sqrt[n]{a_n} \geqq 1$ *for every* n, *in which case* $\sum a_n$ *is divergent.*

For, if $L\sqrt[n]{a_n} = \lambda$, then if r is chosen between λ and 1 we can find ν such that, by the definition of limit,

$$\left|\sqrt[n]{a_n} - \lambda\right| < r - \lambda$$

for $n > \nu$, and therefore $\sqrt[n]{a_n} < r < 1$, hence by the preceding theorem, $\sum a_n$ must be convergent.

If $\lambda > 1$, then $\sqrt[n]{a_n} = \lambda + \delta_n$, where $\delta_n \rightarrow 0$, and for n large enough, say $n > \nu$, we have $\lambda + \delta_n > 1$, and therefore $\sqrt[n]{a_n} > 1$, and $\sum a_n$ must be divergent.

If $\lambda = 1$, but $\sqrt[n]{a_n} \geqq 1$, then $a_n \geqq 1$ and a_n cannot $\rightarrow 0$, hence $\sum a_n$ must be divergent.

If $\lambda = 1$, but $\sqrt[n]{a_n} < 1$, there is no test.

This second form of the radical test is usually easier to apply than the preceding form.

Examples.—(1) $\sum_1^\infty \frac{1}{n^n}$ is convergent since $\sqrt[n]{a_n} = \frac{1}{n} \rightarrow 0$.

(2) $\sum_2^\infty \frac{1}{(\log n)^n}$ is convergent since $\sqrt[n]{a_n} = \frac{1}{\log n} \rightarrow 0$.

(3) $\sum \dfrac{n^{n^2}}{(n+1)^{n^2}}$ is convergent since

$$\sqrt[n]{a_n} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \rightarrow \frac{1}{e} < 1.$$

The radical test may be put in the more general form:

Theorem.—*If* $\overline{L}\sqrt[n]{a_n} = \omega$, *then* $\sum a_n$ *is convergent when* $\omega < 1$, *and is divergent when* $\omega > 1$; *the case* $\omega = 1$ *remains in doubt. Also if* $\underline{L}\sqrt[n]{a_n} > 1$, *the series is divergent.*

For, if $\omega < 1$, let $1 - \alpha$ be a number between ω and 1; then by definition of upper limit, there exist only a finite number of terms of the sequence $(\sqrt[n]{a_n})$ which are $>1 - \alpha$, hence a positive integer N can be found such that $\sqrt[n]{a_n} < 1 - \alpha$ for every $n > N$. Hence, by the first form of the radical test, the series $\sum a_n$ is convergent.

If $\omega > 1$, let $1 + \alpha$ be a number between 1 and ω, then there are an infinite number of terms of the sequence $(\sqrt[n]{a_n})$ which are $> 1 + \alpha$, and hence there are an infinite number of values of n for which $a_n > 1$. Therefore, $\sum a_n$ is divergent.

If $\underline{L}\sqrt[n]{a_n} > 1$, a constant g exists such that for $n > m$, we have $\sqrt[n]{a_n} > g > 1$, and hence $\sum a_n$ is divergent.

104. Relation between the Ratio and Radical Tests.—By Art. **50**, we find that when the ratio test applies to a given series, the radical test must also apply, but not conversely. The radical test is more general than the ratio test, since it may apply in cases where the ratio test is not applicable; but in practice the ratio test is usually easier to apply.

105. Theorem.—*If* $\overline{L}n^k a_n < \infty \, (k > 1)$, *then* $\sum a_n$ *is convergent, and if* $\underline{L}\, n a_n > 0$, *then* $\sum a_n$ *is divergent.*

This test is obtained by comparison with the hyperharmonic series. For, if $\overline{L}n^k a_n < \infty$, there must exist a constant $g > 0$ such that for every n, $n^k a_n < g$, or

$$a_n < g \cdot \frac{1}{n^k}.$$

Then each term of $\sum a_n$ is $<$ the corresponding term of the convergent series $\sum \frac{1}{n^k}$ $(k > 1)$, so that $\sum a_n$ is convergent.

If $\underline{L} na_n > 0$, there must exist a constant h such that for every n, $na_n > h$; or

$$a_n > h \cdot \frac{1}{n},$$

and each term of $\sum a_n$ is $>$ the corresponding term of the divergent series $\sum \frac{1}{n}$, hence $\sum a_n$ is divergent.

Examples.—(1) For the series

$$\sum \frac{1}{(\log n)^m} \quad (m>0),$$

we have

$$na_n = \frac{n}{(\log n)^m} \to \infty \text{ (Art. } \mathbf{60}\text{)},$$

hence the series is divergent.

(2) For the series $\sum \frac{1}{n \log n}$, we have

$$na_n = \frac{1}{\log n} \to 0,$$

so that the test of Art. **105** does not apply. The series is, however, shown to be divergent in Art. **113**.

(3) For the series

$$\sum \frac{1}{n^\alpha (\log n)^\beta} \ (\beta \text{ arbitrary}),$$

we have

$$na_n = \frac{n^{1-\alpha}}{(\log n)^\beta} \to \infty \text{ for } \alpha < 1 \text{ (Art. } \mathbf{60}\text{)}.$$

Hence, the series is divergent for $\alpha < 1$.

(4) In the series $\sum \frac{1}{\sqrt[n]{n^{n+1}}}$, we have

$$na_n = \frac{1}{\sqrt[n]{n}} \to 1 \quad \text{(Art. } \mathbf{51}\text{)},$$

hence the series is divergent.

106. Theorem. *If* $\overline{L}a_nC_n < \infty$, *and if* $\sum \frac{1}{C_n}$ *is convergent, then* $\sum a_n$ *is convergent. If* $\underline{L}a_nD_n > 0$, *and if* $\sum \frac{1}{D_n}$ *is divergent, then* $\sum a_n$ *is divergent.*

For, in the first case we can find a constant g such that $a_nC_n < g$, or $a_n < g \cdot \frac{1}{C_n}$, so that $\sum a_n$ is convergent.

It is not necessary to assume the existence of the limit La_nC_n.

The test is sufficient, but not necessary. This may be shown by taking $C_n = n!$, and $a_n = \frac{1}{2^{n-1}}$, for $a_nC_n > \frac{n}{2}$, so that $La_nC_n = \infty$; but $\sum a_n$ is convergent.

If $\underline{L}a_nD_n > 0$, we can find a constant h such that $a_nD_n > h$, or $a_n > h \cdot \frac{1}{D_n}$, so that $\sum a_n$ is divergent.

Here, also, we need not assume the existence of La_nD_n, and the test is sufficient, but not necessary.

107. Kummer's Test. Theorem.—*Let* $k_1, k_2, k_3, \ldots, k_n, \ldots,$ *be any sequence of positive numbers, and put* $K_n \equiv k_n \cdot \frac{a_n}{a_{n+1}} - k_{n+1}$, *then* $\sum a_n$ *is convergent if* $K_n \geqq k > 0$ *for every* n, *and is divergent if* $K_n \leqq 0$ *for every* n *and* $\sum \frac{1}{k_n}$ *is divergent.*

For, if $K_n \geqq k$, we obtain

$$k_m a_m - k_{m+1} a_{m+1} \geqq k a_{m+1},$$

since $a_{n+1} > 0$; putting $m = 1, 2, 3, \ldots, n - 1$ in succession, we get

$$\begin{aligned} ka_2 &\leqq k_1a_1 - k_2a_2, \\ ka_3 &\leqq k_2a_2 - k_3a_3, \\ ka_4 &\leqq k_3a_3 - k_4a_4, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ ka_n &\leqq k_{n-1}a_{n-1} - k_na_n. \end{aligned}$$

Adding these inequalities, we have

$$ka_2 + ka_3 + \ . \ . \ . + ka_n \leqq k_1a_1 - k_na_n,$$

or

$$kA_n \leqq k_1a_1 + ka_1 - k_na_n < ka_1 + k_1a_1,$$

$$\therefore A_n < a_1\left(1 + \frac{k_1}{k}\right).$$

Hence, by Art. **80,** $\sum a_n$ is convergent.

If $K_n \leqq 0$, we have $\dfrac{a_n}{a_{n+1}} \leqq \dfrac{k_{n+1}}{k_n}$, or $\dfrac{a_{n+1}}{a_n} \geqq \dfrac{1}{k_{n+1}} \div \dfrac{1}{k_n}$, and since $\sum \dfrac{1}{k_n}$ is divergent, we may apply Art. **100,** and we see that $\sum a_n$ is divergent.

The numbers k_n are called **Kummer's numbers,** and the series $\sum \dfrac{1}{k_n}$ **Kummer's series.**

Kummer's test is often more useful in the form:

Theorem.—*If* LK_n *exists* $= K$, *then* $\sum a_n$ *is convergent if* $K > 0$, *and is divergent if* $K < 0$, and $\sum \dfrac{1}{k_n}$ *is divergent. When* $K = 0$, *there is no test, unless* $K_n < 0$ *for every* n, *in which case* $\sum a_n$ *is divergent.*

This is readily derived from the preceding form.

A more general form of this test is:

Theorem.—$\sum a_n$ *is convergent if* $\underline{L}K_n > 0$, *and is divergent if* $\overline{L}K_n < 0$ *and* $\sum \dfrac{1}{k_n}$ *is divergent.*

Kummer's test includes several other tests as special cases; in particular, the ratio test (Art. **101**) and Raabe's test (Art. **108**). To get the ratio test we take $k_n = 1$.

108. Raabe's Test.—In Kummer's test, take $k_n = n$. Then $\sum \frac{1}{k_n} = \sum \frac{1}{n}$ is divergent and

$$K_n = n\left(\frac{a_n}{a_{n+1}}\right) - \left(n + 1\right) = n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1,$$

and we derive the test:

Theorem.—*$\sum a_n$ is convergent if $n\left(\frac{a_n}{a_{n+1}} - 1\right) \geqq q > 1$ for every n, and is divergent if $n\left(\frac{a_n}{a_{n+1}} - 1\right) \leqq 1$ for every n.*

Using the second form of Kummer's test, we get another form of Raabe's test:

Theorem.—*If $R_n \equiv n\left(\frac{a_n}{a_{n+1}} - 1\right)$, and if $\underset{n}{L} R_n$ exists $= R$, then $\sum a_n$ is convergent if $R > 1$, and is divergent if $R < 1$. There is no test when $R = 1$, unless $R_n \leqq 1$ for every n, in which case $\sum a_n$ is divergent.*

Raabe's test is used when the ratio test fails, when $L \frac{a_n}{a_{n+1}} = 1$.

Examples.—(1) The series

$$A = \frac{2}{4} + \frac{2 \cdot 3}{4 \cdot 5} + \frac{2 \cdot 3 \cdot 4}{4 \cdot 5 \cdot 6} + \cdot \cdot \cdot$$

is convergent since

$$R_n = n\left(\frac{n + 4}{n + 2} - 1\right) = n \cdot \frac{2}{n + 2} = \frac{2}{1 + \frac{2}{n}} \rightarrow 2 > 1.$$

The ratio test fails here.

(2) For the hyperharmonic series $\sum \frac{1}{n^2}$, we have

$$\frac{a_n}{a_{n+1}} = \frac{(n + 1)^2}{n^2} = \left(1 + \frac{1}{n}\right)^2 \rightarrow 1,$$

but

$$R_n = n\left(\frac{(n+1)^2}{n^2} - 1\right) = \frac{(n+1)^2 - n^2}{n} = \left(2 + \frac{1}{n}\right) \to 2 > 1,$$

hence $\sum \frac{1}{n^2}$ is convergent by this test. The general case of this series for any k may be treated similarly.

(3) $\sum \frac{1}{n^2 - n}$ is convergent since $R_n = \frac{2n^2}{n^2 - n} \to 2 > 1$.

The more general form of Raabe's test is:

Theorem.—*The series* $\sum a_n$ *is convergent if* $\underline{L}R_n > 1$, *and is divergent if* $\overline{L}R_n < 1$.

Raabe's test may also be stated in a somewhat different form, as follows:

Theorem.—*If* $\frac{a_{n+1}}{a_n}$ *can be put in the form*

$$\frac{a_{n+1}}{a_n} = \frac{1}{1 + \frac{\alpha_n}{n}},$$

where $L\frac{\alpha_n}{n} = 0$, *then if as n increases,* α_n *finally becomes and remains greater than some number which is* >1, *the series* $\sum a_n$ *is convergent, but if* α_n *finally becomes and remains* < 1, *the series is divergent.*

We see that $\alpha_n = R_n$.

We may prove this form of the test directly as follows: Suppose $\alpha_n > 1 + \alpha$, $\alpha > 0$, for $n \geqq \nu$. Then

$$\frac{a_{n+1}}{a_n} = \frac{1}{1 + \frac{\alpha_n}{n}} < \frac{1}{1 + \frac{1 + \alpha}{n}} \quad (n \geqq \nu).$$

This reduces to

$$a_{n+1} < \frac{1}{\alpha}[na_n - (n+1)a_{n+1}], \quad (n \geqq \nu).$$

In this inequality substitute in succession $\nu, \nu + 1, \ . \ . \ . \ ,$ $\nu + p$ for n, and add the resulting inequalities, and we obtain

$$a_{\nu+1} + a_{\nu+2} + \ . \ . \ . \ + a_{\nu+p} < \frac{1}{\alpha}\left[\nu a_\nu - (\nu + p)a_{\nu+p}\right] < \frac{\nu a_\nu}{\alpha}.$$

This shows that the sum of the first p terms of the series $\sum\limits_{m=\nu+1}^{\infty} a_m$ is always less than the constant $\frac{\nu a_\nu}{\alpha}$, hence by Art. **80,** the series is convergent, and therefore $\sum\limits_1^\infty a_n$ is convergent.

If, however, $\alpha_n < 1$ for $n > \nu$, then

$$\frac{a_{n+1}}{a_n} = \frac{1}{1 + \frac{\alpha_n}{n}} > \frac{1}{1 + \frac{1}{n}} \ (n > \nu).$$

But $\frac{1}{1 + \frac{1}{n}}$ is the ratio $\frac{d_{n+1}}{d_n}$ for the series $\sum d_n = \sum \frac{1}{n}$, hence by Art. **100** the series $\sum a_n$ is divergent.

Example.—For the series for which $\frac{a_{n+1}}{a_n} = \frac{2n}{2n + 3}$, we have

$$\frac{a_{n+1}}{a_n} = \frac{1}{1 + \frac{3}{2n}} = \frac{1}{1 + \frac{3/2}{n}},$$

so that $\alpha_n = 3/2 > 1$, and $\sum a_n$ is convergent.

109. Dini's Theorem.—*If* $\sum a_n$ *is divergent,* $\sum \frac{a_n}{A_n}$ *is also divergent.*

For,

$$\frac{a_{m+1}}{A_{m+1}} + \frac{a_{m+2}}{A_{m+2}} + \ . \ . \ . \ + \frac{a_{m+p}}{A_{m+p}}$$
$$> \frac{1}{A_{m+p}}(a_{m+1} + a_{m+2} + \ . \ . \ . \ a_{m+p})$$
$$> \frac{A_{m+p} - A_m}{A_{m+p}} = 1 - \frac{A_m}{A_{m+p}}.$$

Now let m remain fixed and let $p \to \infty$, then since $A_{m+p} \to \infty$, we have

$$\frac{L}{p}\left(\frac{a_{m+1}}{A_{m+1}} + \frac{a_{m+2}}{A_{m+2}} + \ . \ . \ . \ + \frac{a_{m+p}}{A_{m+p}}\right) \geqq 1,$$

and hence $\sum \frac{a_n}{A_n}$ must be divergent.

Examples.—If we take $\sum a_n = 1 + 1 + 1 + \ . \ . \ .$, we get the divergent series $\sum \frac{a_n}{A_n} = \sum \frac{1}{n}$, the harmonic series. If we take $\sum a_n = \sum \frac{1}{n}$, we find $\sum \frac{a_n}{A_n} = \sum \frac{1}{nH_n}$, which must be divergent, then $\sum \frac{1}{nH_{n-1}}$ must also be divergent; but $H_{n-1} > \log n$, and $\frac{1}{nH_{n-1}} < \frac{1}{n \log n}$, and hence $\sum \frac{1}{n \log n}$ must be divergent.

110. Cauchy's Condensation Test. Theorem.—*If* $\phi(n)$ *is a positive monotonic decreasing function of* n, *and if* a *is any positive integer, then the series*

$$(a) \qquad \phi(1) + \phi(2) + \phi(3) + \ . \ . \ . \ + \phi(n) + \ . \ . \ .$$

and

$$(b) \quad a\phi(a) + a^2\phi(a^2) + a^3\phi(a^3) + \ . \ . \ . \ + a^n\phi(a^n) + \ . \ . \ .$$

are both convergent or both divergent.

In the first series consider the terms

$$(c) \quad \phi(a^k + 1),\ \phi(a^k + 2),\ \phi(a^k + 3),\ . \ . \ . \ ,\ \phi(a^{k+1}).$$

The number of these terms is $a^{k+1} - a^k = a^k(a - 1)$, and since each of them (except the last) is greater than $\phi(a^{k+1})$, their sum is

$$> a^k(a - 1)\phi(a^{k+1}), \text{ or } > \frac{a-1}{a} a^{k+1}\phi(a^{k+1}).$$

By giving to k in succession the values 0, 1, 2, 3, . . . , we get

$$\phi(2) + \phi(3) + \phi(4) + \ . \ . \ . \ + \phi(a) > \frac{a-1}{a} \cdot a\phi(a),$$
$$\phi(a+1) + \phi(a+2) + \ . \ . \ . \ + \phi(a^2) > \frac{a-1}{a} a^2\phi(a^2).$$
. . . .

If S_n and S'_n denote the sum of n terms[1] of the series (a) and (b) respectively, we get by addition,

$$S_n - \phi(1) > \frac{a-1}{a} \cdot S'_n,$$

hence if the second series (b) is divergent, so also is the first, (a).

Again, each term of (c) is $< \phi(a^k)$, and the sum of these terms is $< (a-1)a^k\phi(a^k)$. Giving k the values 0, 1, 2, 3, . . . , in succession, we get

$$\phi(2) + \phi(3) + \ . \ . \ . \ + \phi(a) < (a-1)\phi(1),$$
$$\phi(a+1) + \phi(a+2) + \ . \ . \ . \ + \phi(a^2) < (a-1)a\phi(a),$$
. . . .

By addition, we get

$$S_n - \phi(1) < (a-1)[S'_n + \phi(1)],$$

hence if series (b) is convergent, so also is the first.

Hence, the series $\sum \phi(n)$, $\sum a^n\phi(a^n)$ converge or diverge simultaneously.

Example.—Let us apply this test to the series $\sum \frac{1}{n(\log n)^p}$. This series will be convergent or divergent for the same values of p as the series whose general term is

$$a^n \cdot \frac{1}{a^n(\log a^n)^p} = \frac{1}{(n \log a)^p} = \frac{1}{(\log a)^p} \cdot \frac{1}{n^p}.$$

---

[1] n is some function of a.

The constant factor $\frac{1}{(\log a)^p}$ is common to every term, hence the given series is convergent at the same time as $\sum \frac{1}{n^p}$, and hence is convergent for $p > 1$, and divergent for $p \leq 1$.

111. Maclaurin's Integral Test.[1] **Theorem.**—*If* $\phi(x)$ *is a positive monotonic decreasing function which* $\rightarrow 0$ *as* $x \rightarrow \infty$ *in the interval* (a, ∞), *then the series* $\sum_{n=1}^{\infty} \phi(n)$ *is convergent when* $\underset{l \rightarrow \infty}{L} \int_a^l \phi(x)\, dx$ *exists (finite), and is divergent when* $\underset{l \rightarrow \infty}{L} \int_a^l \phi(x)\, dx = \infty$.

If we draw the curve $y = \phi(x)$, we see that this curve approaches the X-axis as an asymptote. Consider the set of rectangles starting with the ordinate $x = a$, with the base $= 1$, and altitudes $\phi(a)$, $\phi(a + 1)$, . . . , $\phi(a + n)$. Each rectangle extends beyond the curve $y = \phi(x)$, so that the sum of the areas of the rectangles is $>$ the area under the curve, that is, the area between the curve and the X-axis and the two ordinates $x = a$ and $x = a + n$, so that we have

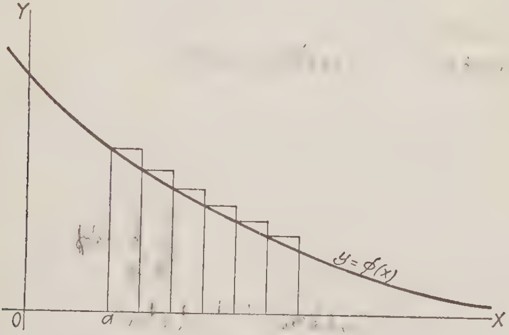

FIG. 4.

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[1] Often incorrectly called Cauchy's integral test.

$$(a)\quad \phi(a) + \phi(a+1) + \ . \ . \ . \ + \phi(a+n-1) > \int_a^{a+n} \phi(x)dx.$$

Now construct a set of rectangles under the curve, with bases = 1, and altitudes $\phi(a+1)$, $\phi(a+2)$, . . . , $\phi(a+n)$, then the sum of the areas of the rectangles is evidently < the area under the curve, so that

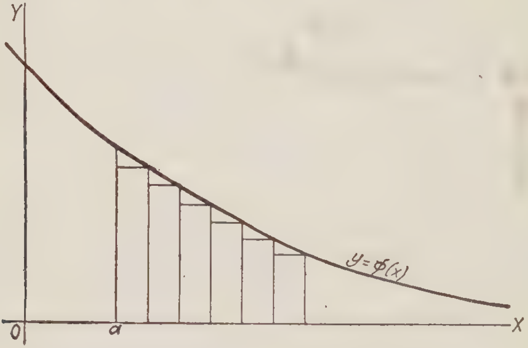

FIG. 5.

$$(b)\quad \phi(a) + \phi(a+1) + \ . \ . \ . \ + \phi(a+n) < \phi(a) + \int_a^{a+n} \phi(x)dx.$$

If the integral $\int_a^{a+n} \phi(x)dx$ approaches a limit as $n \to \infty$, from (b) we see that $\sum_{n=a}^{\infty} \phi(n)$ is convergent (Art. **80**), and therefore $\sum_{1}^{\infty} \phi(n)$ is convergent when $\underset{l}{L}\int_a^l \phi(x)dx$ exists with a finite value.

If the integral $\int_a^{a+n} \phi(x)dx \to \infty$ as $n \to \infty$, then by (a) the sum $\sum_{m=a}^{a+n} \phi(m) \to \infty$, and hence $\sum_{n=1}^{\infty} \phi(n)$ is divergent when $\underset{l}{L}\int_a^l \phi(x)dx = \infty$.

This test may be proved analytically without the use of geometric considerations, as follows:

If x is in the interval $(n, n + 1)$, where $n \geqq m \geqq a$, then

$$\phi(n + 1) \leqq \phi(x) \leqq \phi(n),$$

$$\int_n^{n+1} \phi(n + 1)dx \leqq \int_n^{n+1} \phi(x)dx \leqq \int_n^{n+1} \phi(n)dx,$$

or

$$\phi(n + 1) \leqq \int_n^{n+1} \phi(x)dx \leqq \phi(n).$$

Put $n = m, m + 1, m + 2, \ . \ . \ . \ , m + p$ in succession, and add, and we get

$$(c) \quad \phi(m + 1) + \phi(m + 2) + \ . \ . \ . \ + \phi(m + p + 1) \leqq \int_m^{m+p} \phi(x)dx$$
$$\leqq \phi(m) + \phi(m + 1) + \ . \ . \ . \ + \phi(m + p).$$

Now if $\underset{l}{L}\int_a^l \phi(x)dx$ exists, as $p \to \infty$ the integral

$$\int_m^{m+p} \phi(x)dx$$

will approach a definite limit, and from the left-hand inequality of (c), we see that $\sum_{m+1}^{\infty} \phi(n)$, and therefore $\sum_{1}^{\infty} \phi(n)$ must converge. But if $\underset{l}{L}\int_a^l \phi(x)dx = \infty$, then as $p \to \infty$, $\int_m^{m+p} \phi(x)dx \to \infty$, and therefore from the right-hand inequality of (c), we see that $\sum_{1}^{\infty} \phi(m)$ will diverge.

Example.—Consider the hyperharmonic series $\sum \frac{1}{n^k}$. Here $\phi(x) = \frac{1}{x^k}$, $a = 1$. This function evidently satisfies the conditions of the integral test. We have

$$\underset{l}{L}\int_1^l \frac{dx}{x^k} = \underset{l}{L}\left\{\frac{1}{1 - k} \cdot \frac{1}{l^{k-1}} - \frac{1}{1 - k}\right\},$$

hence $\underset{l}{L} \int_1^l \frac{dx}{x^k}$ is finite when $k > 1$, and is ∞ when $k \leqq 1$, so that $\sum \frac{1}{n^k}$ is convergent when $k > 1$, and divergent when $k \leqq 1$, as we have already learned by other methods.

112. Iterated Logarithms.—For the iterated logarithms $\log (\log x)$, $\log [\log (\log x)]$, etc., it is usual to use a special abbreviated notation. We put

$$l_0x \equiv x,\ l_1x \equiv \log x,\ l_2x \equiv \log (\log x) = \log l_1x,$$
$$l_3x \equiv \log \log \log x = \log l_2x,\ \ldots,\ l_rx = \log l_{r-1}x.$$

113. Bertrand's Series. Theorem.—*The series*

$$\sum \frac{1}{n(l_1n)^p},\ \sum \frac{1}{nl_1n(l_2n)^p},\ \sum \frac{1}{nl_1nl_2n(l_3n)^p},$$
$$\ldots,\ \sum \frac{1}{nl_1nl_2n\ \ldots\ l_{r-1}n \cdot (l_rn)^p},\ \ldots$$

are all convergent when $p > 1$ and divergent when $p \leqq 1$.

Since $\log x$ does not exist in the system of real numbers for $x \leqq 0$, the summation in the above series must begin with a value of n for which l_rn exists.

Let us consider first the series

$$\sum \frac{1}{n(l_1n)^p} \text{ or } \sum \frac{1}{n(\log n)^p} \quad \text{for } p > 1.$$

Since

$$\frac{d}{dx} (\log x)^{1-p} = \frac{1-p}{x(\log x)^p},$$

we have

$$\int_a^l \frac{dx}{x(\log x)^p} = \frac{1}{p-1}\left[\frac{1}{(\log a)^{p-1}} - \frac{1}{(\log l)^{p-1}}\right],\ 0 < a < l,$$
$$\therefore \int_a^\infty \frac{dx}{x(\log x)^p} = \frac{1}{p-1}\ \frac{1}{(\log a)^{p-1}},$$

and since this value is finite, the series must be convergent for $p > 1$.

Now if $p = 1$, we have

$$\frac{dl_2x}{dx} = \frac{d}{dx}\log(\log x) = \frac{1}{x\log x},$$

so that

$$\int_a^l \frac{dx}{x\log x} = \log(\log l) - \log(\log a),$$

hence $\int_a^l \frac{dx}{x\log x} \to \infty$ as $l \to \infty$, and the series is divergent for $p = 1$. By Art. **92,** we see that the series will be divergent for $p < 1$.

The other series may now be treated in similar manner; for the general case we would use the function

$$\phi(x) = \frac{1}{xl_1xl_2x \quad . \quad . \quad . \quad l_{r-1}x(l_rx)^p}$$

in the integral test.

The series

$$\sum\frac{1}{n(l_1n)^p}, \sum\frac{1}{nl_1n(l_2n)^p}, \sum\frac{1}{nl_1nl_2n(l_3n)^p}, \quad . \quad . \quad . \quad ,$$

are called **Bertrand's series,** also the **logarithmic scale.** We may apply this scale as a test of convergence. We compare the terms of a given series with the first series of the scale, and if this fails to give a test, we compare with the next series of the scale, etc.

114. Theorem.—*Let*

$$\mathfrak{L}_0(n) \equiv n\left(\frac{a_n}{a_{n+1}} - 1\right) = R_n,$$

$$\mathfrak{L}_1(n) \equiv l_1n\left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right],$$

$$\mathfrak{L}_2(n) \equiv l_2n\left\{l_1n\left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] - 1\right\},$$

$$\mathfrak{L}_3(n) \equiv l_3n\left(l_2n\left\{l_1n\left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] - 1\right\} - 1\right),$$

. . . .

If there exists an r *such that* $\underset{n}{L}\mathfrak{L}_r(n)$ *exists* $= \mathfrak{L}_r$, *then* $\sum a_n$ *is convergent if* $\mathfrak{L}_r > 1$ *and divergent if* $\mathfrak{L}_r < 1$. *There is no test for* $\mathfrak{L}_r = 1$.

This result is obtained from Kummer's test by taking

$$k_n = nl_1n,\ nl_1nl_2n,\ nl_1nl_2nl_3n,\ \ldots$$

115. Logarithmic Criteria. Theorem.—*If the expression*

$$\frac{\log \dfrac{1}{a_n}}{\log n}$$

is always greater than a fixed number which is > 1, *the positive series* $\sum a_n$ *is convergent; but if that expression is always* < 1, *the series* $\sum a_n$ *is divergent.*

For if

$$\frac{\log \dfrac{1}{a_n}}{\log n} > k > 1,$$

we have

$$\log \frac{1}{a_n} > k \log n, \text{ or } a_n < \frac{1}{n^k}.$$

But $\sum \dfrac{1}{n^k}$ is convergent for $k > 1$, hence $\sum a_n$ is convergent.

If

$$\frac{\log \dfrac{1}{a_n}}{\log n} < 1,$$

we have

$$\log \frac{1}{a_n} < \log n, \text{ or } a_n > \frac{1}{n},$$

and since $\sum \dfrac{1}{n}$ is divergent, $\sum a_n$ is also divergent.

This test may also be formulated:

Theorem.—*If* $\underset{n}{L}\dfrac{\log\dfrac{1}{a_n}}{\log n}$ *exists* $= l$, *then* $\sum a_n$ *is convergent if* $l > 1$, *and divergent if* $l < 1$, *and no test if* $l = 1$.

116. Bertram's Tests. Theorem.—*If*

$$Q_s(n) = \frac{\log \dfrac{1}{a_n \cdot nl_1nl_2n\ \ .\ \ .\ \ .\ \ l_{s-1}n},}{l_{s+1}n}$$

then $\sum a_n$ *is convergent if for some* s *and* m,

$$Q_s(n) \geqq \mu > 1,\ n > m,$$

and is divergent if

$$Q_s(n) \leqq 1.$$

For, multiplying $Q_s(n) \geqq \mu > 1$ by $l_{s+1}n$, we get

$$l_{s+1}n \cdot Q_s(n) \geqq l_{s+1}n \cdot \mu,$$

or

$$\log \frac{1}{a_n \cdot nl_1n\ \ .\ \ .\ \ .\ \ l_{s-1}n} \geqq \log\ (l_sn)^\mu.$$

Hence

$$\frac{1}{a_n \cdot nl_1n\ \ .\ \ .\ \ .\ \ l_{s-1}n} \geqq (l_sn)^\mu,$$

or

$$a_n \leqq \frac{1}{nl_1n\ \ .\ \ .\ \ .\ \ l_{s-1}n \cdot (l_sn)^\mu}.$$

Thus $\sum a_n$ is convergent by comparison with Bertrand's series. The rest of the theorem follows similarly.

117. Cahen's Test. Theorem.—*If for every* n,

$$C_n \equiv n\left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] \leqq K(const.),$$

then $\sum a_n$ *is divergent.*

For,

$$\mathfrak{L}_1(n) = \frac{\log n}{n} \cdot C_n \leqq \frac{\log n}{n} \cdot K,$$

so that $\mathcal{L}_1(n) \to 0$, and by Art. **114,** $\sum a_n$ is divergent.

This test may also be stated in the form:

Theorem.—*For* $\sum a_n$ *to converge, it is necessary that*

$$\overline{L}n\left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] = +\infty.$$

It may also be put in a form corresponding to the last form of Raabe's test:

Theorem.—*If* $\dfrac{a_{n+1}}{a_n} = \dfrac{1}{1 + \dfrac{\alpha_n}{n}}$ *and* $\alpha_n = 1 + \dfrac{\beta_n}{n}$, *where* $\dfrac{\beta_n}{n} \to 0$, *then* $\sum a_n$ *is divergent if* β_n *remains* $<$ *some constant* K.

For $\beta_n = C_n$. This result may also be proved directly as follows. We have

$$\frac{a_{n+1}}{a_n} = \frac{1}{1 + \dfrac{1}{n} + \dfrac{\beta_n}{n^2}} > \frac{1}{1 + \dfrac{1}{n} + \dfrac{K}{n^2}}.$$

Now the series $\sum \dfrac{1}{n - K}$ is divergent (compare with the harmonic series), and the ratio $\dfrac{d_{n+1}}{d_n}$ for this series is

$$\frac{1}{(n+1) - K} \div \frac{1}{n - K} = \frac{n - K}{n - K + 1} = \frac{1}{1 + \dfrac{1}{n - K}} < \frac{1}{1 + \dfrac{1}{n} + \dfrac{K}{n^2}},$$

since

$$\frac{1}{n - K} = \frac{1}{n} \cdot \frac{1}{1 - \dfrac{K}{n}} = \frac{1}{n} + \frac{K}{n^2} + \frac{K^2}{n^3} + \dots$$

Hence

$$\frac{a_{n+1}}{a_n} > \frac{1}{1 + \frac{1}{n} + \frac{K}{n^2}},$$

which is > the ratio $\frac{d_{n+1}}{d_n}$ of a known divergent series, therefore $\sum a_n$ is divergent.

Example.—For the series for which $\frac{a_{n+1}}{a_n} = \frac{3n^3 - 2n^2}{3n^3 + n^2 + 1}$, we have

$$\alpha_n = \frac{3n^2 + 1}{3n^2 - 2n}, \quad \beta_n = \frac{2n + 1}{3n - 2} \rightarrow \frac{2}{3},$$

hence $\sum \alpha_n$ is divergent.

118. Gauss' Test. Theorem.—*If*

$$\frac{a_n}{a_{n+1}} = \frac{n^p + \alpha_1 n^{p-1} + \ . \ . \ . \ + \alpha_p}{n^p + \beta_1 n^{p-1} + \ . \ . \ . \ + \beta_p},$$

where $p, \alpha_1, \ . \ . \ . \ , \alpha_p, \beta_1, \ . \ . \ . \ , \beta_p$ *are independent of* n, *then* $\sum a_n$ *is convergent if* $\alpha_1 - \beta_1 > 1$, *and divergent if* $\alpha_1 - \beta_1 \leqq 1$.

For, we find

$$R_n = n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{(\alpha_1 - \beta_1) + \frac{1}{n}(\alpha_2 - \beta_2 + \ . \ . \ .)}{1 + \frac{1}{n}(\beta_1 + \ . \ . \ . \ )}$$

$$\rightarrow \alpha_1 - \beta_1.$$

Hence, by Art. **114**, $\sum a_n$ is convergent when $\alpha_1 - \beta_1 > 1$, and divergent when $\alpha_1 - \beta_1 < 1$.

If $\alpha_1 - \beta_1 = 1$, Raabe's test does not apply, but we can use Cahen's test. We have $LC_n = \alpha_2 - \beta_2 - \beta_1$, and by Art. **117** $\sum a_n$ must be divergent.

Examples.—(1) $\sum \frac{1\cdot 3\cdot 5.\ \ .\ 2n-1}{2\cdot 4\cdot 6.\ \ .\ 2n}\cdot\frac{1}{2n+1}$ is convergent, by Gauss' rule, for

$$\frac{a_n}{a_{n+1}} = \frac{4n^2+2n}{4n^2-4n+1} = \frac{n^2+\frac{1}{2}n}{n^2-n+\frac{1}{4}},$$

$$\beta_1 = -1,\ \alpha_1 = \tfrac{1}{2} > \beta_1 + 1.$$

(2) $\sum \frac{3\cdot 5\ .\ .\ .\ 2n+1}{2\cdot 4\ .\ .\ .\ 2n}\cdot\frac{1}{2n+3}$ is divergent since

$$\frac{a_n}{a_{n+1}} = \frac{2n}{2n+1}\cdot\frac{2n+3}{2n+1} = \frac{4n^2+6n}{4n^2+4n+1} = \frac{n^2+\frac{3}{2}n}{n^2+n+\frac{1}{4}},$$

$$\beta = 1,\ \alpha_1 = \tfrac{3}{2} < \beta_1 + 1.$$

(3) The hypergeometric series

$$1 + \frac{\alpha\cdot\beta}{1\cdot\gamma} + \frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot 2\cdot\ \ \gamma(\gamma+1)} + \frac{\alpha(\alpha+1)(\alpha+2)\,.\,\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\ \ \gamma(\gamma+1)(\gamma+2)} + \ .\ .\ .$$

is convergent, by Gauss' test, when $\alpha+\beta-\gamma<0$, for

$$\frac{a_{n+1}}{a_{n+2}} = \frac{n^2+n(1+\gamma)+\gamma}{n^2+n(\alpha+\beta)+\alpha\beta}.$$

119. Theorem.—*If* $\frac{a_n}{a_{n+1}} = 1 + \frac{\alpha}{n} + \frac{\beta_n}{n^\mu}$, $\mu > 1$, β_n *finite, then* $\sum a_n$ *is convergent if* $\alpha > 1$ *and divergent if* $\alpha \leqq 1$.

For,

$$\mathfrak{L}_0(n) = \alpha + \frac{\beta_n}{n^{\mu-1}} \to \alpha,$$

since $\mu > 1$, and by Art. **108,** $\sum a_n$ is convergent if $\alpha > 1$ and divergent if $\alpha < 1$. If $\alpha = 1$, we cannot apply Art. **108,** but Art. **114** can be used. We have

$$\mathfrak{L}_1(n) = \log n\,[\mathfrak{L}_0(n)-1] = \frac{\log n}{n^{\mu-1}}\cdot\beta_n \to 0 \text{ (Art. \textbf{59})},$$

hence $\sum a_n$ is divergent.

120. Rapidity of Convergence and Divergence.—As usual, let A_n, B_n denote the sum of the first n terms of the positive series $\sum a_n$, $\sum b_n$, and $\overline{A}_n$, $\overline{B}_n$ the remainders after n terms in the same series. Then, if $\sum a_n$, $\sum b_n$ are both divergent, we say that $\sum a_n$ is **more slowly divergent** than $\sum b_n$ if $A_n \prec B_n$, *i.e.*, if $L\dfrac{A_n}{B_n} = 0$; and we say that $\sum a_n$ is **more rapidly divergent** than $\sum b_n$ if $A_n \succ B_n$, *i.e.*, if $L\dfrac{A_n}{B_n} = \infty$; while if $A_n \sim gB_n$, *i.e.*, if $L\dfrac{A_n}{B_n} = g \neq 0, < \infty$, we say that $\sum a_n$ and $\sum b_n$ **diverge equally fast.**

If $\sum a_n$, $\sum b_n$ are both convergent, we say that $\sum a_n$ is **more rapidly convergent** than $\sum b_n$ if $\overline{A}_n \prec \overline{B}_n$, or $L\dfrac{\overline{A}_n}{\overline{B}_n} = 0$; and we say that $\sum a_n$ is **more slowly convergent** than $\sum b_n$ if $\overline{A}_n \succ \overline{B}_n$, or $L\dfrac{\overline{A}_n}{\overline{B}_n} = \infty$; while they are said to **converge equally fast** if $\overline{A}_n \sim g\overline{B}_n$, *i.e.*, if $L\dfrac{\overline{A}_n}{\overline{B}_n} = g \neq 0$, $< \infty$.

121. Theorem.—*If the ratio of the general terms of two positive series approaches zero, the first series is more slowly divergent or more rapidly convergent than the second.*

Let $L\dfrac{a_n}{b_n} = 0$, and suppose first that $\sum b_n$ is divergent. Since $B_n \rightarrow \infty$, we have by the Cauchy-Stolz theorem,

$$L\frac{A_n}{B_n} = L\frac{A_n - A_{n-1}}{B_n - B_{n-1}} = L\frac{a_n}{b_n} = 0.$$

$\sum a_n$ is therefore more slowly divergent than $\sum b_n$ by definition. The series might possibly be convergent in this case.

Now if $\sum b_n$ is convergent, we know the same to be true of $\sum a_n$, since $\frac{a_n}{b_n}$ remains finite (Art. **94**). Since $\overline{A}_n$ and $\overline{B}_n \rightarrow 0$, we have

$$L\frac{\overline{A}_n}{\overline{B}_n} = L\frac{\overline{A}_n - \overline{A}_{n-1}}{\overline{B}_n - \overline{B}_{n-1}} = L\frac{a_n}{b_n} = 0,^1$$

and therefore $\sum a_n$ is more rapidly convergent than $\sum b_n$.

122. Theorem.—*If a positive series $\sum a_n$ is given, we can always construct another which will converge more slowly or diverge more slowly.*

(1) If $\sum a_n$ converges with sum A, and we take

$$b_1 = \sqrt{A} - \sqrt{\overline{A}_1},\ b_2 = \sqrt{\overline{A}_1} - \sqrt{\overline{A}_2},\ b_3 = \sqrt{\overline{A}_2} - \sqrt{\overline{A}_3},\ \ldots,$$

then the series $\sum b_n$ is convergent, since the sum of n terms of $\sum b_n$ is $B_n = \sqrt{A} - \sqrt{\overline{A}_n}$, and therefore $B = \sqrt{A}$ in the limit. Since the remainder $\overline{B}_n = B - B_n = \sqrt{A} - B_n = \sqrt{\overline{A}_n}$, we have $L\frac{\overline{A}_n}{\overline{B}_n} = L\sqrt{\overline{A}_n} = 0$, hence $\sum b_n$ is more slowly convergent than $\sum a_n$.

(2) If $\sum a_n$ diverges, we take

$$b_1 = \sqrt{A_1},\ b_2 = \sqrt{A_2} - \sqrt{A_1},\ b_3 = \sqrt{A_3} - \sqrt{A_2},\ \ldots,$$

then $\sum b_n$ is divergent, since $B_n = \sqrt{A_n} \rightarrow \infty$, then $L\frac{B_n}{A_n} = L\frac{1}{\sqrt{A_n}} = 0$, so that $\sum b_n$ is more slowly divergent than $\sum a_n$.

It is impossible to assign a boundary between convergent and divergent series. In other words, if two series are given, the one convergent and the other divergent, we

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[1] Using a theorem analogous to the Cauchy-Stolz theorem, but in which $u_n \rightarrow 0$, $v_n \rightarrow 0$. (See PIERPONT, "Theory of Functions of Real Variables," vol. I, art. **184**.)

can set up infinitely many series which converge more slowly than the given convergent series, and infinitely many series which diverge more slowly than the given divergent series.

123. Tests of Convergence for Series of Positive and Negative, or Complex Terms.—Corresponding to every test for the convergence of positive series, there is a test for the absolute convergence of non-positive series. Thus, if $L\left|\frac{u_{n+1}}{u_n}\right| < 1$, the series $\sum u_n$ is absolutely convergent; and if $L\sqrt[n]{|u_n|} < 1$, $\sum u_n$ is absolutely convergent.

124. Further Developments of the Theory of Convergence and Divergence of Positive Series.—In this chapter we have discussed the simpler and more important criteria of convergence, but a large number of other investigations of tests of convergence and divergence have been made, which cannot be touched upon here. The most thorough and complete treatment of the subject is that of Pringsheim in Math. Annalen, vol. 35, pp. 297–394.

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CHAPTER IX

SERIES OF POSITIVE AND NEGATIVE TERMS

125. Theorem.—*If* $\sum c_n$ *is a convergent positive series, and if the numbers* v_n *are such that* $|v_n| < K$ *(a finite constant) for every* n, *then* $\sum c_n v_n$ *is absolutely convergent.*

For, since $\sum c_n$ is convergent, we have

$$\epsilon > 0,\ \nu,\ \sum_{n+1}^{n+p} c_i < \frac{\epsilon}{K},\ n > \nu,\ p = 1, 2, 3, \ \ldots$$

But

$$\sum_{n+1}^{n+p} |c_i v_i| \leq \sum_{n+1}^{n+p} c_i |v_i| < K \sum_{n+1}^{n+p} c_i < K \cdot \frac{\epsilon}{K} = \epsilon,$$

for $n > \nu$, $p = 1, 2, 3, \ldots$. Therefore, by Art. **78,** $\sum c_n v_n$ is absolutely convergent.

Corollary.—If we take $v_n = \pm 1$, we get the theorem already derived in Art. **83**: A series is convergent if the series of its absolute values is convergent.

For if we take c_n of Art. **125** as the absolute value of the u_n of Art. **83,** and $v_n = \dfrac{u_n}{c_n} = \pm 1$, the theorem of Art. **125** applies.

126. Theorem.—*An absolutely convergent series* $\sum u_n$ *remains absolutely convergent if each term* u_n *is multiplied by a factor* v_n *such that* $|v_n| < K$ *for every* n.

For if $a_n = |u_n|$, then $\sum a_n$ is convergent, and we have

$$\epsilon > 0,\ \nu,\ \sum_{n+1}^{n+p} a_i < \frac{\epsilon}{K},\ n > \nu,\ p = 1, 2, 3, \ \ldots .$$

But

$$\sum_{n+1}^{n+p} |u_i v_i| = \sum_{n+1}^{n+p} a_i |v_i| < K \cdot \sum_{n+1}^{n+p} a_i < K \cdot \frac{\epsilon}{K} = \epsilon$$

for $n > \nu$, $p = 1, 2, 3, \ldots$ Hence, $\sum |u_n v_n|$ is convergent, and therefore $\sum u_n v_n$ is absolutely convergent.

127. Theorem.—*If* $\sum d_n$ *is a divergent positive series, and* $v_n \geqq G$ *(a positive constant) for every* n, *then* $\sum d_n v_n$ *is divergent.*

For

$$\sum_{n=1}^{m} d_n v_n \geqq G \sum_{n=1}^{m} d_n,$$

but $\sum_{1}^{m} d_n \to \infty$ as $m \to \infty$, therefore $\sum_{1}^{m} d_n v_n \to \infty$, and $\sum d_n v_n$ is divergent.

Or we may apply Art. **125.** If we assume $\sum d_n v_n$ to be convergent, then $\sum d_n$ can be obtained from $\sum d_n v_n$ by multiplying its terms by $\frac{1}{v_n}$, which satisfies the conditions of Art. **125,** so that $\sum d_n$ would be convergent, contrary to hypothesis.

128. Abel's Theorem.—*If* $\sum u_n$ *is a convergent series, and the sequence* (v_n) *is bounded and monotonic, then* $\sum u_n v_n$ *is convergent.*

For by Abel's lemma (Art. **42**),

$$\left| \sum_{n+1}^{n+p} u_i v_i \right| < \overline{B}\, V,$$

where B is the upper bound of $|U_{n+1}|$, $|U_{n+2}|$, . . . , $|U_{n+p}|$, and V is the upper bound of $|v_{n+i}|$. Since $\sum u_n$ is convergent, we can find ν such that $B < \frac{\epsilon}{V}$ for $n > \nu$, hence

$$\left|\sum_{n+1}^{n+p} u_i v_i\right| < \frac{\epsilon}{V} \cdot V = \epsilon,\ n > \nu,\ p = 1, 2, 3, \ . \ . \ .$$

Therefore, $\sum u_n v_n$ is convergent.

129. Dirichlet's Theorem.—*If* $\sum u_n$ *is an oscillating series, and* (v_n) *is a decreasing sequence which tends to* 0 *as a limit, then* $\sum u_n v_n$ *is convergent.*

For, by Abel's lemma,

$$\left|\sum_{n+1}^{n+p} u_i v_i\right| < \overline{B} \cdot v_{n+1},$$

where $\overline{B}$ is the upper bound of $|U_{n+1}|$, $|U_{n+2}|$, . . . , $|U_{n+p}|$; $\overline{B}$ exists, finite, since $\sum u_n$ is oscillating. But $v_{n+1} \longrightarrow 0$ as $n \longrightarrow \infty$, so that $v_{n+1} < \frac{\epsilon}{\overline{B}}$ for $n > \nu$, hence

$$\left|\sum_{n+1}^{n+p} u_i v_i\right| < \overline{B} \cdot \frac{\epsilon}{\overline{B}} = \epsilon,\ n > \nu,\ p = 1, 2, 3, \ . \ . \ . \ .$$

Therefore, $\sum u_n v_n$ is convergent

Example.—The series

$$\sin\theta + \sin 2\theta + \ . \ . \ . + \sin n\theta + \ . \ . \ .$$

is convergent or oscillating; for if $\sin\theta = 0$, every term of the series is 0, and if $\sin\theta \neq 0$, the sum of the first n terms, by a formula of Trigonometry, is

$$\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} \cdot \sin\frac{n+1}{2} \cdot \theta,$$

which is less in absolute value than

$$\left|\frac{1}{\sin\frac{\theta}{2}}\right|,$$

and hence the series is either convergent or oscillating. Now taking the sequence (v_n) to be $\left(\frac{1}{n}\right)$, we see that the series

$$\sin\theta + \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} + \ .\ .\ .\ + \frac{\sin n\theta}{n} + \ .\ .\ .$$

is convergent for all values of θ.

Corollary.—If we take for the oscillating series $\sum u_n$ the series $1 - 1 + 1 - 1 + \ .\ .\ .\ ,$ we get the alternating series test of Art. **81.**

130. Theorem.—*If* $\sum u_n$ *is convergent, and* (v_n) *is bounded, and* $\sum |v_{n+1} - v_n|$ *is convergent, then* $\sum u_n v_n$ *is convergent.*

For, since $\sum u_n$ is convergent, ν can be found such that

$$|U_{n,i}| < \frac{\epsilon}{K}, n > \nu, i = 1, 2, 3, \ .\ .\ .\ , p, \ .\ .\ .\ ,$$

where K is some finite constant. Then by Abel's identity (Art. **41**),

$$\left|\sum_{n+1}^{n+p} u_i v_i\right| \leqq \sum_{n+1}^{n+p-1} |U_{n,i}|\cdot|v_{i+1} - v_i| + |U_{n,p}|\cdot|v_{n+p}|$$
$$< \frac{\epsilon}{K}\left\{|v_{n+2} - v_{n+1}| + \ .\ .\ .\ + |v_{n+p}|\right\}, n > \nu.$$

Since $\sum |v_{i+1} - v_i|$ is convergent, we have

$$\left\{|v_{n+2} - v_{n+1}| + |v_{n+3} - v_{n+2}| + \ .\ .\ .\ + |v_{n+p}|\right\} < K,$$

where K is a finite constant, hence

$$\left|\sum_{n+1}^{n+p} u_i v_i\right| < \frac{\epsilon}{K}\cdot K = \epsilon, n > \nu, p = 1, 2, 3, \ .\ .\ .\ ,$$

Therefore, $\sum u_n v_n$ is convergent.

Abel's theorem (Art. **128**) may be derived from this. For if (v_n) is monotonic, $(v_{n+1} - v_n)$ is always either positive or negative, and $\sum |v_{n+1} - v_n|$ is telescopic, and

$$\sum_1^n |v_{i+1} - v_i| = (v_2 - v_1) + (v_3 - v_2) + \dots + (v_{n+1} - v_n) = v_{n+1} - v_1,$$

or

$$= (v_1 - v_2) + (v_2 - v_3) + \dots + (v_n - v_{n+1}) = v_1 - v_{n+1},$$

and since (v_n) is bounded and monotonic, $v_{n+1} \to$ a limit, hence $\sum |v_{n+1} - v_n|$ is convergent, and the preceding theorem applies.

131. Theorem.—*If* $\sum u_n$ *is oscillating, and* $\sum |v_{n+1} - v_n|$ *is convergent, and* $v_n \to 0$, *then* $\sum u_n v_n$ *is convergent.*

For, by Abel's identity,

$$\left|\sum_{n+1}^{n+p} u_i v_i\right| \leqq \sum_{n+1}^{n+p-1} |U_{n,\,i}| \cdot |v_i - v_{i+1}| + |U_{n,\,p}| \cdot |v_{n+p}|;$$

since $\sum u_n$ is oscillating, every $|U_{n.i}| < K$ (some finite constant), and

$$\left|\sum_{n+1}^{n+p} u_i v_i\right| < K \left\{ \sum_{n+1}^{n+p-1} |v_i - v_{i+1}| + |v_{n+p}| \right\}.$$

Since $\sum |v_{n+1} - v_n|$ is convergent, and $v_n \to 0$, an index ν exists such that

$$\{|v_{n+1} - v_{n+2}| + |v_{n+2} - v_{n+3}| + \dots + |v_{n+p}|\} < \frac{\epsilon}{K},\ n > \nu,$$

hence

$$\left|\sum_{n+1}^{n+p} u_i v_i\right| < K \cdot \frac{\epsilon}{K} = \epsilon,\ n > \nu,\ p = 1, 2, 3, \ldots,$$

therefore $\sum u_n v_n$ is convergent.

Dirichlet's theorem (Art. **129**) can be derived from this in just the same way as Abel's theorem was derived in Art. **130**.

132. Theorem.—*If* $\sum u_n$ *is any real convergent series, and if* $\sum \frac{1}{d_n}$ *is a divergent positive series, then if* $Ld_n u_n$ *exists, it must be* $= 0$.

For, suppose $Ld_n u_n > 0$, then after a certain value of n, say $n > p$, we should have $d_n u_n > 0$. We can regard the series $\sum u_n$ as being derived from $\sum \frac{1}{d_n}$ by multiplying its terms by the numbers $d_n u_n$, and since these numbers are all positive for $n > p$, the series $\sum u_n$ would have its terms ultimately all positive. The numbers $d_n u_n$ would then ultimately all be $>$ a fixed positive number, hence by Art. **127**, the series $\sum u_n$ would be divergent. Similar considerations hold if we assume $Ld_n u_n < 0$. Therefore, if $Ld_n u_n$ exists, it must have the value 0. It may happen, however, that $d_n u_n$ does not approach any limit, but oscillates.

In particular, if $d_n = n$, we have the

Theorem.—*If* $\sum u_n$ *is convergent, and if* Lnu_n *exists, it must have the value* 0.

The condition $Lnu_n = 0$ is neither necessary nor sufficient for convergence. That it is not sufficient is shown by the fact that there are divergent series for which it is satisfied, as for example, in $\sum \frac{1}{nH_n}$, or $\sum \frac{1}{n \log n}$, and that it is not necessary is shown by the fact that there are convergent series for which nu_n has no limit, as for example,

$$1 - 1/2 + 1/3 - 1/4 + \ldots$$

133. Theorem.—*If the terms of the positive series* $\sum a_n$ *are monotonic decreasing, the condition* $Lna_n = 0$ *is necessary for convergence.*

For if $\sum a_n$ is convergent, a number m can be found such that

$$a_{m+1} + a_{m+2} + \ldots + a_n < \epsilon,$$

and since the terms form a monotonic decreasing sequence, each of these terms is $\leqq a_m$, and therefore $(n - m)a_n < \frac{\epsilon}{2}$ for $n > m$. Now since $a_n \to 0$, N can be found such that $ma_n < \frac{\epsilon}{2}$ for $n > N > m$. From $(n - m)a_n < \frac{\epsilon}{2}$ and $ma_n < \frac{\epsilon}{2}$, we get by addition, $na_n < \epsilon$ for $n > N$, hence $Lna_n = 0$.

The theorem may also be proved by use of the Cauchy-Stolz theorem. Put

$$u_n = \frac{A_n}{a_n} - n \qquad v_n = \frac{1}{a_n},$$

then we have

$$\frac{u_n}{v_n} = A_n - na_n, \qquad \frac{u_n - u_{n-1}}{v_n - v_{n-1}} = A_{n-1},$$

and since $A_n \to A$, $A_{n-1} \to A$, we get

$$A = A - Lna_n, \text{ or } Lna_n = 0.$$

134. Theorem.—*The series* $\sum_1^\infty \left(1 - \frac{1}{n} \log n\right)^n$ *is divergent.*

Put $p = \frac{n}{\log n}$, then

$$\log na_n = \left\{1 + p \log\left(1 - \frac{1}{p}\right)\right\} \log n.$$

Since

$$\frac{1}{p} < -\log\left(1 - \frac{1}{p}\right) = \log\left(1 + \frac{1}{p-1}\right) < \frac{1}{p-1},$$

we have

$$0 < -\log na_n < \frac{\log n}{p-1}.$$

Now if $n \to \infty$, p also $\to \infty$, and hence

$$L\frac{\log n}{p-1} = L\frac{p}{p-1} \cdot \frac{\log^2 n}{n} = 0,$$

therefore $L \log na_n = 0$ and $Lna_n = 1$. But by Art. **132** if $Lna_n \neq 0$, the series cannot be convergent, hence the given series is divergent.

135. Jamet's Test for Convergence of Positive Series.—*The positive series* $\sum a_n$ *is convergent if as* n *increases indefinitely, the expression*

$$(a) \qquad \left(1 - \sqrt[n]{a_n}\right)\frac{n}{\log n}$$

finally remains greater than a number which is itself greater than unity, and is divergent if this expression finally never exceeds unity.

Put $a_n = n^{-p_n}$, then

$$\sqrt[n]{a_n} = n^{-\frac{p_n}{n}} = e^{-\frac{p_n}{n}\log n} > 1 - \frac{p_n}{n}\log n,$$

(Art. **55**), hence

$$p_n > \left(1 - \sqrt[n]{a_n}\right)\frac{n}{\log n}.$$

If the expression (a) is always $>p > 1$, the same is true of p_n, and therefore the series $\sum a_n$ is convergent since its terms are less than those of the known convergent series $\sum \frac{1}{n^p}$. If the expression (a) finally remains $\leqq 1$,

$$\left(1 - \sqrt[n]{a_n}\right)\frac{n}{\log n} \leqq 1,$$

or

$$a_n \geqq \left(1 - \frac{1}{n}\log n\right)^n.$$

But the series $\sum \left(1 - \frac{1}{n} \log n\right)^n$ is divergent (Art. **134**), hence $\sum a_n$ must be divergent since its terms are greater than those of a divergent series.

In applying Jamet's test, it is often more convenient to consider the limit of expression (a) if such exists. Thus we get the

Theorem.—*If* $L\left(1 - \sqrt[n]{a_n}\right)\frac{n}{\log n} = J$, *then* $\sum a_n$ *is convergent if* $J > 1$, *and divergent if* $J < 1$, *and there is no test if* $J = 1$.

When the test fails for $J = 1$, we can consider expressions of the form

$$\left[\left(1 - \sqrt[n]{a_n}\right)\frac{n}{\log n} - 1\right]\frac{\log n}{\log \log n},$$

making use of the fact that $\sum_2^\infty \frac{1}{n(\log n)^q}$ converges for $q > 1$, and diverges for $q \leqq 1$.

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CHAPTER X

CONDITIONAL AND UNCONDITIONAL CONVERGENCE

136. Rearrangement of Terms.—Let $\sum u_n$ be any given series. If $\sum v_n$ has each of its terms identical with a definite term of $\sum u_n$ and conversely; that is, if each v_n is identical with some $u_{n'}$, where n and n' are each finite, then $\sum v_n$ is said to be a **rearrangement** of $\sum u_n$. The rearrangement must be according to some law or norm.

Example.—The series

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \cdots$$

is a rearrangement of

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots$$

137. Conditional and Unconditional Convergence.—*If the convergence of a series is not changed by* **every** *rearrangement of terms, the series is said to be* **unconditionally convergent.** *If a convergent series becomes non-convergent by some rearrangement of terms, it is called* **conditionally convergent.**

138. Theorem.—*A convergent positive series is unconditionally convergent; that is, the convergence of a positive series is not affected by any rearrangement of terms, and the sum of the rearranged series is the same as that of the original series.*

Let $\sum a_n$ be the given series, and $\sum a'_n$ any rearrangement of terms of $\sum a_n$, and put $A'_{n'} \equiv a'_1 + a'_2 + \cdots + a'_{n'}$. Then since the positive series $\sum a_n$ is convergent with sum A, we have

$$\epsilon > 0,\ \nu,\ A - A_n < \epsilon,\ n > \nu,$$

and since the terms of $\sum a_n$ are all positive, it follows that

$$A - A_{n+p} < \epsilon, n > \nu, p = 1, 2, 3, \ldots$$

Now a number n' can be found such that the first n terms of $\sum a_n$ are contained in the sum of the first n' terms of $\sum a'_{n'}$, so that $A_n \leqq A'_{n'}$. Also a number n'' can be found such that the first n' terms of $\sum a'_{n'}$ are all contained in the sum of the first n'' terms of $\sum a_n$, so that $A_n \leqq A'_{n'} \leqq A_{n''}$. Since $A - A_n < \epsilon$, $A - A_{n''} < \epsilon$, $(n'' = n + p)$, we have $A - A'_{n'} < \epsilon$ for every $n' > \nu$. Therefore $A'_{n'} \rightarrow A$ as $n \rightarrow \infty$, and hence $\sum a'_{n'}$ is convergent and has the sum A.

139. Theorem.—*A divergent positive series remains properly divergent after every rearrangement of terms.*

For with the same notation as in Art. **138**, we have $A'_{n'} \geqq A_n$; but by hypothesis, $A_n \rightarrow \infty$ as $n \rightarrow \infty$, and since $n' \geqq n$ and $A'_{n'} \geqq A_n$, we have $A'_{n'} \rightarrow \infty$, and hence $\sum a'_{n'}$ is properly divergent.

140. Dirichlet's Theorem.—*An absolutely convergent series is unconditionally convergent; that is, an absolutely convergent series remains absolutely convergent after every rearrangement of terms and its sum remains the same.*

For, if $\sum u_n$ is absolutely convergent, and (a_n) and $(-b_n)$ are the positive and negative terms of $\sum u_n$, then by Art. **85**, $\sum a_n$ and $\sum b_n$ are convergent. But by Art. **138**, a rearrangement of terms in the positive series $\sum a_n$ and $\sum b_n$ does not affect their convergence or sums, hence $\sum u_n$ will remain convergent and its sum will remain unaltered.

141. Riemann's Theorem.—*The terms of a simply convergent series can always be rearranged so that,* (1) *the new* *series is convergent with any arbitrarily given number as sum, or* (2) *so that the new series is divergent, or* (3) *so that the new series oscillates between arbitrarily assigned bounds; and each of these rearrangements may be made in infinitely many ways. That is, a simply convergent series is conditionally convergent.*

The first part of this proposition may be proved as follows. Suppose we wish to make the sum of the rearranged series equal to σ. We write down the positive terms in order, and stop at the first term which makes the sum of these terms $> \sigma$; this can be done since by Art. **85** the series $\sum a_n$ and $\sum b_n$ are divergent; next we write down the negative terms, and stop at the first term which makes the sum $< \sigma$; we continue the process by adding on further positive terms, stopping at the first term which makes the sum $> \sigma$; and so on. The values of the sums thus obtained oscillate about σ, and the range through which the oscillation takes place diminishes continually, tending to a limit 0, since the general term of a convergent series $\rightarrow 0$. Hence, the new series converges to the sum σ.[1]

142. Example of Riemann's Theorem.—Consider the simply convergent series

$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdot \cdot \cdot + \frac{1}{2n+1} - \frac{1}{2n+2} + \cdot \cdot \cdot$$

The sum of this series is

$$\underset{m}{L} \sum_{n=0}^{m} \left(\frac{1}{2n+1} - \frac{1}{2n+2} \right).$$

If we write the terms of the series in another order, putting two negative terms after each positive term, we get

$$B = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdot \cdot \cdot$$
$$+ \frac{1}{2n+1} - \frac{1}{4n+2} - \frac{1}{4n+4} + \cdot \cdot \cdot$$

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[1] For a more detailed proof, see HOBSON, "Theory of Functions of a Real Variable," art. **334**.

It can be easily shown from a study of the sums A_{3n}, A_{3n+1}, and A_{3n+2} that the new series converges. Its sum is

$$\underset{m}{L} \sum_{n=0}^{m} \left(\frac{1}{2n+1} - \frac{1}{4n+2} - \frac{1}{4n+4} \right).$$

But we have the identity

$$\frac{1}{2n+1} - \frac{1}{4n+2} - \frac{1}{4n+4} = \frac{1}{2} \left(\frac{1}{2n+1} - \frac{1}{2n+2} \right),$$

so that the sum of the second series is half the sum of the first series.

Next consider the rearrangement

$$C = 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \cdots$$

We have

$$\begin{aligned} C_{3n} &= 1 + \frac{1}{3} + \cdots + \frac{1}{4n-1} - \frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2n} \\ &= H_{4n} - \frac{1}{2} H_{2n} - \frac{1}{2} H_n \\ &= (H_{4n} - H_{2n}) + \frac{1}{2} (H_{2n} - H_n) \\ &= A_{4n} + \frac{1}{2} A_{2n}. \end{aligned}$$

If we now let $n \to \infty$, we obtain

$$C = A + \tfrac{1}{2}A = \tfrac{3}{2}A.$$

143. Dirichlet's Theorem.—*For the unconditional convergence of a series, it is necessary and sufficient that it converge absolutely.*

For, if $\sum u_n$ be absolutely convergent, then it is unconditionally convergent by Art. **140**, so that the absolute convergence is a sufficient condition for unconditional convergence. If $\sum u_n$ is unconditionally convergent, it must be absolutely convergent, for if it were not, it would be conditionally convergent by Art. **141**, and hence the condition is necessary.

We have seen that the concepts of absolute convergence and unconditional convergence are coextensive and for all practical purposes identical; similarly with simple convergence and conditional convergence. From now on we shall use only the terms absolute convergence and conditional convergence, as is the usual practice.

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CHAPTER XI

MULTIPLICATION OF SERIES

144. Multiplication of Series.—If we have two infinite series $\sum u_n$ and $\sum v_n$, and wish to form their product, we are naturally led to form the sum $\sum\limits_{i,j} u_iv_j$ for all combinations of i, j without repetition. We may write the first part of this sum:

$$\begin{array}{rcccccc}
\sum\limits_{i,j} u_iv_j = & u_1v_1 & + u_1v_2 & + u_1v_3 + & \cdots & + u_1v_m & + \cdots \\
& + u_2v_1 & + u_2v_2 & + u_2v_3 + & \cdots & + u_2v_m & + \cdots \\
& + u_3v_1 & + u_3v_2 & + u_3v_3 + & \cdots & + u_3v_m & + \cdots \\
& \cdot & \cdot & \cdot\quad\cdot & & \cdot & \cdot \\
& \cdot & \cdot & \cdot\quad\cdot & & \cdot & \cdot \\
& \cdot & \cdot & \cdot\quad\cdot & & \cdot & \cdot \\
& + u_nv_1 & + u_nv_2 & + u_nv_3 + & \cdots & + u_nv_m & + \cdots \\
& \cdot & \cdot & \cdot\quad\cdot & & \cdot & \cdot \\
& \cdot & \cdot & \cdot\quad\cdot & & \cdot & \cdot \\
& \cdot & \cdot & \cdot\quad\cdot & & \cdot & \cdot
\end{array}$$

We shall evidently obtain all terms if we write this in the form:

$$\begin{aligned}
u_1v_1 &+ (u_1v_2 + u_2v_1) + (u_1v_3 + u_2v_2 + u_3v_1) \\
&+ (u_1v_4 + u_2v_3 + u_3v_2 + u_4v_1) + \cdots \\
&+ (u_1v_n + u_2v_{n-1} + \cdots + u_nv_1) + \cdots,
\end{aligned}$$

in which each parenthesis represents the sum of the terms in diagonals of the rectangular array. This may be written more briefly $\sum\limits_1^\infty w_n$, where $w_n = u_1v_n + u_2v_{n-1} + \cdots +$

$u_n v_1$. This series $\sum w_n$ is called the **Cauchy-product** of the two series $\sum u_n$, $\sum v_n$, or simply the **product series.**

145. Mertens' Theorem.—*If* $\sum u_n$, $\sum v_n$ *are convergent with sums* U, V, *and at least one of them, say* $\sum u_n$, *is absolutely convergent, then the product series* $\sum w_n$ *is convergent, and its sum is* $W = UV$.

We have

$$\begin{aligned} w_1 &= u_1 v_1, \\ w_2 &= u_1 v_2 + u_2 v_1, \\ w_3 &= u_1 v_3 + u_2 v_2 + u_3 v_1, \\ w_4 &= u_1 v_4 + u_2 v_3 + u_3 v_2 + u_4 v_1, \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ w_n &= u_1 v_n + u_2 v_{n-1} + \quad . \quad . \quad . \quad + u_n v_1. \end{aligned}$$

Adding, we get

$$W_n = u_1 V_n + u_2 V_{n-1} + u_3 V_{n-2} + \quad . \quad . \quad . \quad + u_n V_1.$$

Substituting $V_n = V - \overline{V}_n$, we get

$$\begin{aligned} W_n &= u_1(V - \overline{V}_n) + u_2(V - \overline{V}_{n-1}) + \quad . \quad . \quad . \quad + u_n(V - \overline{V}_1) \\ &= U_n V - \sigma_n, \end{aligned}$$

where

$$\sigma_n = u_1 \overline{V}_n + u_2 \overline{V}_{n-1} + \quad . \quad . \quad . \quad + u_n \overline{V}_1$$

If we could now prove that $\sigma_n \to 0$, we should have

$$W = LW_n = LU_n V = V \cdot LU_n = UV,$$

and our theorem would be proved.

Let us then seek to prove $L\sigma_n = 0$. Divide n into two parts ν and $n - \nu$, each of which increases indefinitely with n. Then the remainders $\overline{V}_i$ are separated into two sets:

$$\overline{V}_1, \overline{V}_2, \overline{V}_3, \quad . \quad . \quad . \quad , \overline{V}_{n-\nu-1},$$

and

$$\overline{V}_{n-\nu},\ \overline{V}_{n-\nu+1},\ \ldots,\ \overline{V}_n.$$

Since $\sum v_n$ is convergent, this sequence of $\overline{V}_i$'s tends to 0, and therefore the sequence is bounded; let $\overline{B}$ be the upper bound of the absolute values of the first set and $\overline{B}'$ the upper bound of the absolute values of the second set. $\overline{B}$ remains finite as $n \to \infty$, and $\overline{B}' \to 0$ since $n - \nu \to \infty$ as $n \to \infty$. Then

$$(a)\quad |\sigma_n| \leq \overline{B}'(|u_1| + \ldots + |u_\nu|) + \overline{B}(|u_{\nu+1}| + \ldots + |u_n|).$$

Since $\sum u_n$ is absolutely convergent, $\sum |u_n|$ is convergent with sum U' say, and we have

$$|u_1| + |u_2| + \ldots + |u_\nu| < U',$$

and by the general principle of convergence, we have

$$\epsilon > 0,\ N, (|u_{\nu+1}| + |u_{\nu+2}| + \ldots + |u_n|) < \frac{\epsilon}{2\overline{B}},\ \nu > N,$$

since $\overline{B}$ is finite; and since $\overline{B}' \to 0$ as $n \to \infty$ and $\nu \to \infty$, we have

$$\epsilon > 0,\ N',\ \overline{B}' < \frac{\epsilon}{2U'},\ \nu > N',\ n > \nu;$$

$$\therefore |\sigma_n| < \frac{\epsilon}{2U'} \cdot U' + \overline{B} \cdot \frac{\epsilon}{2\overline{B}} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\ \nu > N_1,\ n > \nu,$$

where $N_1 > N, N'$. Hence, $L\sigma_n = 0$, and our proof is complete.

146. Cauchy's Theorem.—*If* $\sum u_n$, $\sum v_n$ *are both absolutely convergent, then the product series* $\sum w_n$ *is absolutely convergent.*

For, since $\sum u_n$, $\sum v_n$ are absolutely convergent, the series $\sum |u_n|$, $\sum |v_n|$ are convergent, and by Art. **145** their product series is convergent, so that if

$$\omega_n = |u_1| \cdot |v_n| + |u_2| \cdot |v_{n-1}| + \ . \ . \ . \ + |u_n| \cdot |v_1|,$$

then $\sum \omega_n$ is convergent. But

$$|w_n| \leqq |u_1| \cdot |v_n| + \ . \ . \ . \ + |u_n \cdot |v_1| = \omega_n,$$

so that by the comparison test, $\sum |w_n|$ is convergent, and therefore $\sum w_n$ is absolutely convergent.

147. Abel's Theorem.—*If* $\sum u_n$, $\sum v_n$ *are convergent with sums* U, V, *and if the product series* $\sum w_n$ *is also convergent with sum* W, *then* $W = UV$.

As in Art. **145**, we have

$$W_n = u_1 V_n + u_2 V_{n-1} + \ . \ . \ . \ + u_n V_1,$$

$$\therefore W_1 + W_2 + \ . \ . \ . \ + W_n = U_1 V_n + U_2 V_{n-1} + \ . \ . \ . \ + U_n V_1,$$

or

$$(a) \quad \frac{1}{n}(W_1 + W_2 + \ . \ . \ . \ + W_n) = \frac{1}{n}(U_1 V_n + U_2 V_{n-1} + \ . \ . \ . \ + U_n V_1).$$

Since $LW_n = W$, we have by Art. **39**,

$$L\frac{1}{n}(W_1 + W_2 + \ . \ . \ . \ + W_n) = W,$$

and since $LU_n = U$ and $LV_n = V$, we have by Art. **40**,

$$L\frac{1}{n}(U_1 V_n + \ . \ . \ . \ + U_n V_1) = UV.$$

Hence, by (a), we get $W = UV$.

148. Multiplication of Conditionally Convergent Series.—When neither $\sum u_n$ nor $\sum v_n$ converges absolutely, the product series may not even converge. This is shown by the following example due to Cauchy.

The series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ . \ . \ .$ is convergent by the alternating series test, but it is not absolutely convergent since $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ . \ . \ .$ is divergent, being the hyperharmonic series for $k = \frac{1}{2}$. If we multiply this series by itself, we get the series

$$1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right)$$
$$- \left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}}\right) + \ . \ . \ .$$
$$\pm \left(\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{n-2}} + \frac{1}{\sqrt{3}}\frac{1}{\sqrt{n-3}} + \ . \ . \ . + \right.$$
$$\left. \frac{1}{\sqrt{n-2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{n-1}}\right) \mp \ . \ . \ .$$

The general term of this series may be written

$$w_n = \pm \sum_{m=1}^{n-1} \frac{1}{\sqrt{m(n-m)}}.$$

By Algebra, $\sqrt{m(n-m)} \leq \frac{n}{2}$, and $\frac{1}{\sqrt{m(n-m)}} \geq \frac{2}{n}$, and $|w_n| > \frac{2(n-1)}{n} = 2\left(1 - \frac{1}{n}\right)$. Hence, $\sum w_n$ cannot be convergent since its n^{th} term does not → 0.

149. Division of Series. Theorem.—*If* $\sum u_n$ *and* $\sum v_n$ *are convergent with sums* U *and* V *respectively, and if* $\sum v_n$ *converges absolutely and* $v_1 \neq 0$, *then*

$$\frac{U}{V} = \frac{\sum u_n}{\sum v_n} = \sum x_n,$$

where

$$x_n = \frac{u_n - x_1v_n - x_2v_{n-1} - \cdots - x_{n-1}v_2}{v_1},$$

provided $\sum x_n$ *is absolutely convergent.*

For, in order that we may have

$$\frac{\sum u_n}{\sum v_n} = \sum x_n,$$

the series $\sum u_n$ must be the product of $\sum v_n$ and $\sum x_n$. Since both $\sum v_n$ and $\sum x_n$ converge absolutely, by hypothesis, they may be multiplied (Art. **146**) and the product written

$$x_1v_1 + (x_1v_2 + x_2v_1) + \cdots + (x_1v_n + x_2v_{n-1} + \cdots + x_nv_1) + \cdots$$

If this series is to be identical with $\sum u_n$, the values of $x_1, x_2, \ldots, x_n, \ldots$ must be determined so that the corresponding terms of the two series are equal; then

$$\begin{aligned}
u_1 &= x_1v_1,\\
u_2 &= x_1v_2 + x_2v_1,\\
u_3 &= x_1v_3 + x_2v_2 + x_3v_1,\\
&\cdots\cdots\cdots\cdots\\
u_n &= x_1v_n + x_2v_{n-1} + \cdots + x_nv_1,\\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

Solving for the x_i's, since $v_1 \neq 0$, we get

$$\begin{aligned}
x_1 &= \frac{u_1}{v_1},\\
x_2 &= \frac{u_2 - x_1v_2}{v_1},\\
x_3 &= \frac{u_3 - x_1v_3 - x_2v_2}{v_1},\\
&\cdots\cdots\cdots\cdots\\
x_n &= \frac{u_n - x_1v_n - x_2v_{n-1} - \cdots - x_{n-1}v_2}{v_1},\\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

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CHAPTER XII

SERIES OF VARIABLE TERMS—UNIFORM CONVERGENCE

150. Convergence of Series of Functions.—Let $\sum_{1}^{\infty} u_n(z)$ be a series whose terms are continuous functions of a complex variable z defined in a region R. When we say that this series converges for a value $z = z_0$ in R, we mean that when z_0 is substituted for z, the series of constant terms $\sum_{1}^{\infty} u_n(z_0)$ is convergent; that is, if $\overline{U}_n(z_0) = \sum_{n+1}^{\infty} u_m(z_0)$ be the remainder after n terms, then $|\overline{U}_n(z_0)| < \epsilon$ for $n >$ some index N, however small $\epsilon > 0$ may be.

151. Uniform Convergence.—*A series* $\sum_{1}^{\infty} u_n(z)$, *each of whose terms is a function of* z *defined in a region* R, *is said to be* **uniformly convergent** *in that region if it converges for every value of* z *in* R, *and if, for any arbitrarily small* $\epsilon > 0$, *an index* N, **independent of z,** *can be found such that* $|\overline{U}_n(z)| < \epsilon$ *for every* $n \geqq N$ *and for every value of* z *in* R.

The essential point in this definition is that the index N shall hold for every z in R. For any preassigned value of z for which the series converges it is evident from the definition of ordinary convergence that for a given ϵ, an index N (dependent in general upon the chosen value of z) can be found which will make $|\overline{U}_n(z)| < \epsilon$ for $n \geqq N$. But, in order that the series shall converge uniformly, it is necessary further, that the same index N should satisfy the condition, no matter what value of z be selected in the region R.

Examples.—(1) Consider the series

$$\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \cdot \cdot \cdot ,$$

in which x is a real variable. This series is convergent by the alternating series test. We find

$$|\overline{U}_n(x)| < \frac{1}{x^2+n+1}.$$

Now if this series is to be uniformly convergent, we must be able to find an index, N, independent of x, such that $\frac{1}{x^2+n+1} < \epsilon$ for $n \geqq N$. Solving $\frac{1}{x^2+n+1} < \epsilon$, we find

$$(a) \qquad n > \frac{1}{\epsilon} - 1 - x^2.$$

If for any particular ϵ, we put $N = \frac{1}{\epsilon} - 1$, then for any finite x, we should have the condition satisfied if we take $n > N$. Hence, the series is uniformly convergent for every finite value of x.

(2) Consider next the series

$$U = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdot \cdot \cdot + \frac{x^2}{(1+x^2)^n} + \cdot \cdot \cdot ,$$

where x is a real variable. This being a geometric series, we find, for $x \neq 0$, its sum $U(x) = 1 + x^2$, and remainder $\overline{U}_n(x) = \frac{1}{(1+x^2)^n}$. If we put $|\overline{U}_n(x)| < \epsilon$, we have

$$\frac{1}{(1+x^2)^n} < \epsilon, \text{ or } 1 + x^2 > e^{\frac{1}{n}\log\frac{1}{\epsilon}}$$

If we choose a value $n = N$ which satisfies this inequality for a definite x, and write

$$1 + x^2 > e^{\frac{1}{N}\log\frac{1}{\epsilon}},$$

since the right-hand side of this inequality is > 1, but constant for assigned values of ϵ and N, we can find positive values of x (near 0) which do not satisfy this inequality. Hence, the condition $|\overline{U}_n(x)| < \epsilon$ not being satisfied for all values of x in the interval $(0, 1)$, the series is not uniformly convergent in that interval.

(3) Now let us examine the series

$$\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \cdots \qquad (x \geqq 0).$$

We have

$$u_n(x) = \frac{1}{(n-1)x+1} - \frac{1}{nx+1},$$

$$\therefore\ U_n(x) = 1 - \frac{1}{nx+1}.$$

If $x > 0$,

$$U(x) = LU_n(x) = 1,\ \overline{U}_n(x) = \frac{1}{nx+1},$$

but if $x = 0$,

$$u_n(0) = 0,\ U_n(0) = 0,\ U(0) = 0$$

Now we have $\overline{U}_n(x) < \epsilon$ or $\frac{1}{nx+1} < \epsilon$, provided

$$n > \frac{\frac{1}{\epsilon} - 1}{x}.$$

If x is fixed, however small ϵ, we can find N $\left(\text{the next integer above } \frac{\frac{1}{\epsilon}-1}{x}\right)$ such that $\overline{U}_n(x) < \epsilon$ for $n \geqq N$.

But this value depends on x, and as $x \to 0$, we have $N \to \infty$, and hence the series is not uniformly convergent in the interval (0, 1).

152. Geometrical Discussion of Uniform Convergence.—If the series $\sum u_n(x)$ is uniformly convergent in an interval ($a \leqq x \leqq b$), there exists an index N such that for $n \geqq N$, and for every x in the interval,

$$|U(x) - U_n(x)| < \epsilon,$$

or

$$U_n(x) - \epsilon < U(x) < U_n(x) + \epsilon,\ n \geqq N.$$

Now if we draw the curve $y = U_N(x)$, and the two parallel curves $y = U_N(x) \pm \epsilon$, at a distance ϵ on each side, then the curve $y = U(x)$ must lie within this strip in all the interval. If we make ϵ decrease, the index N cannot decrease but may increase, and we thus bring the two curves bounding the strip closer and closer to one another, and crush up $y = U(x)$ in all the interval (a, b) within two continuous and neighboring curves. Hence, in the case of uniform convergence, the approximation curves $y = U_n(x)$ will give a correct indication of the shape of the curve throughout the interval of uniform convergence.

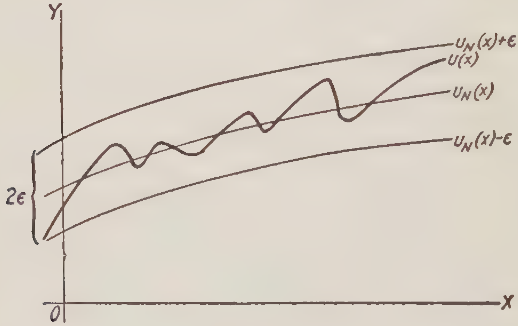

Fig. 6.

153. Weierstrass' M-test for Uniform Convergence. Theorem.—*If the terms of* $\sum u_n(z)$ *are continuous functions of* z *in a region* R, *and if* $|u_n(z)| \leqq M_n$ *for every* n *and for all values of* z *in the region* R, *and the* M_n *are positive constants, independent of* z, *then if* $\sum M_n$ *is convergent, the series* $\sum u_n(z)$ *is uniformly convergent in* R, *and moreover it is also absolutely convergent in* R.

For, since $\sum M_n$ is convergent, its remainder satisfies the condition

$$\sum_{n+1}^{\infty} M_i < \epsilon,\ n \geqq N.$$

But from $|u_i(z)| \leqq M_i$ for every i and every z in R, we get

$$\left|\sum_{n+1}^{\infty} u_i(z)\right| \leqq \sum_{n+1}^{\infty} |u_i(z)| \leqq \sum_{n+1}^{\infty} M_i < \epsilon$$

for $n \geqq N$ and for every z in R, hence $\sum u_n(z)$ is uniformly convergent in R. By Arts. **92, 84,** we see that $\sum u_n(z)$ is also absolutely convergent.

This criterion, due to Weierstrass, gives a very convenient practical test for uniform convergence, and is the one most widely used; it serves for most of the series met with in elementary analysis.

Examples.—(1) The series $\sum M_n \sin nx$ is uniformly convergent in any interval whatever, provided $\sum M_n$ converges, since $|\sin nx| \leqq 1$. Thus the series $\sum_{1}^{\infty} \frac{\sin nx}{n^p}$ $(p > 1)$ converges uniformly for all real values of x, since $\left|\frac{\sin nx}{n^p}\right| \leqq \frac{1}{n^p}$ for every x, and $\sum \frac{1}{n^p}$ is convergent for $p > 1$. Similarly for the series $\sum_{1}^{\infty} \frac{\cos nx}{n^p}$ $(p > 1)$.

(2) $\sum \frac{e^{nx}-1}{2^n e^{nx}}$ is uniformly convergent in the interval $(0, \infty)$, since

$$\frac{e^{nx}-1}{2^n e^{nx}} < \frac{1}{2^n}$$

for every $x > 0$, and $\sum \frac{1}{2^n}$ is convergent.

Weierstrass' test gives a criterion for both uniform and absolute convergence. But uniform convergence of a series does not ensure its absolute convergence, nor conversely. Thus, the series $\sum \frac{x^2}{(1+x^2)^n}$ converges absolutely, but we have seen (in Art. **150**) that it does not converge uniformly near $x = 0$; the series $\sum_1^\infty \frac{(-1)^{n-1}}{x^2+n}$ converges uniformly, but is only conditionally convergent; that it is not absolutely convergent is shown by the series $\sum_1^\infty \frac{1}{|n+x^2|}$ being divergent; for all real values of x the terms of the series are alternately positive and negative, and numerically decreasing, so that by Art. **81** the remainder after n terms is numerically less than the n^{th} term, and hence it follows that the series is uniformly convergent.

154. Tannery's Theorem.—*If*

$$F(n) = v_0(n) + v_1(n) + v_2(n) + \ . \ . \ . \ + v_p(n),$$

wnere p becomes infinite along with n, and if $\underset{n}{L} v_r(n) = w_r$ (r fixed), and $|v_r(n)| \leq M_r$, where M_r is positive and independent of n, and $\sum M_r$ is convergent, then

$$\underset{n}{L} F(n) = w_0 + w_1 + w_2 + \ . \ . \ . \ + w_n + \ . \ . \ . \ = W.$$

Since $\sum M_r$ is convergent, we can choose q so that $\sum_q^\infty M_r < \epsilon$; then choose n so large that $p > q$; we have then

$$|v_q + v_{q+1} + \ . \ . \ . + v_p| \leqq \sum_q^\infty M_r < \epsilon,$$

or

$$|F(n) - (v_0 + v_1 + \ . \ . \ . + v_{q-1})| < \epsilon.$$

We have also

$$|w_q + w_{q+1} + w_{q+2} + \ . \ . \ . + w_p + \ . \ . \ . \text{ to } \infty| \leqq \sum_q^\infty M_r < \epsilon.$$

Then

$$\begin{aligned}|F(n) - \sum_0^\infty w_r| < |(v_0 + v_1 + \ . \ . \ . + v_{q-1})\\ - (w_0 + w_1 + \ . \ . \ . + w_{q-1})| + 2\epsilon.\end{aligned}$$

Since q is fixed and independent of N and $\underset{n}{L} v_r(n) = w_r$, if we let $n \rightarrow \infty$, we find

$$\overline{L}|F(n) - W| \leqq 2\epsilon.$$

Since ϵ is arbitrarily small, we have

$$L[F(n) - W] = 0, \qquad \text{or} \qquad \underset{n}{L} F(n) = W = \sum_0^\infty w_r.$$

155. Continuity. Theorem.—*If* $\sum u_n(z)$ *is uniformly convergent in a region* R, *and its terms* $u_n(z)$ *are continuous functions of* z *in that region, then its sum* $U(z)$ *is itself a continuous function of* z *in* R.

Let z_0 be a value of z in the region R, and let $z_0 + h$ be a value of z in the neighborhood of z_0 which also lies in the region R. Then we may write

$$\begin{aligned}U(z_0 + h) - U(z_0) = U(z_0 + h) - U_n(z_0 + h)\\ + U_n(z_0 + h) - U_n(z_0) + U_n(z_0) - U(z_0),\end{aligned}$$

and therefore

$$|U(z_0 + h) - U(z_0)| \leqq |U(z_0 + h) - U_n(z_0 + h)| \\ + |U_n(z_0 + h) - U_n(z_0)| + |U_n(z_0) - U(z_0)|.$$

Since $\sum u_n(z)$ is uniformly convergent in R, for any arbitrarily assigned small ϵ, we can find an index N such that $|\overline{U}_n(z)| < \frac{\epsilon}{3}$ for $n \geqq N$ and for every z in R, and therefore we have

$$|U(z_0 + h) - U_n(z_0 + h)| < \frac{\epsilon}{3}, \quad |U_n(z_0) - U(z_0)| < \frac{\epsilon}{3},$$

for $n \geqq N$. But since each $u_n(z)$ is continuous and $U_n(z)$ is a sum of a finite number of such continuous functions, we have, by the definition of continuity of functions,

$$|U_n(z_0 + h) - U_n(z_0)| < \frac{\epsilon}{3}$$

for $|h| <$ some δ. We get by combination of the three inequalities,

$$|U(z_0 + h) - U(z_0)| < \epsilon$$

for $|h| < \delta$, which shows that $U(z)$ is continuous for $z = z_0$.

This continuity theorem may be given another form, as follows: In the inequality

$$|U(z_0 + h) - U(z_0)| < \epsilon, \; |h| < \delta,$$

if we put $z_0 + h = z$, we have

$$|U(z) - U(z_0)| < \epsilon, \; |z - z_0| < \delta,$$

which may be written

$$\underset{z \to z_0}{L} \; U(z) = U(z_0),$$

or

$$\underset{z \to z_0}{L} \sum_1^\infty u_n(z) = \sum_1^\infty u_n(z_0) = \sum_1^\infty \underset{z \to z_0}{L} \; u_n(z).$$

Thus we have the

Theorem.—*When a series* $\sum u_n(z)$ *is uniformly convergent in a region* R, *and its terms are continuous functions in that region, then if* z_0 *is a point of* R, *we have*

$$\underset{z \to z_0}{L} \sum_1^\infty u_n(z) = \sum_1^\infty \underset{z \to z_0}{L} u_n(z).$$

156. Theorem.—*A series which represents a discontinuous function cannot be uniformly convergent in a region which includes the point of discontinuity.*

For if the series were uniformly convergent, it would follow from Art. **155** that the function would be continuous in its region of uniform convergence.

Uniform convergence is a *sufficient* condition for continuity but it is not a necessary condition, for non-uniformly convergent series are known which represent continuous functions in the region of non-uniform convergence.

Example.—The series

$$x^2 + \frac{x^2}{1 + x^2} + \frac{x^2}{(1 + x^2)^2} + \cdot \cdot \cdot ,$$

already discussed in Art. **151,** has been shown to be non-uniformly convergent in the neighborhood of $x = 0$, and we shall now show that it is also discontinuous there. We have $U(x) = 1 + x^2$ for $x \neq 0$, and when $x \to 0$, $U(x) \to 1$, but $U(0) = 0$ since when we put $x = 0$ every term of the series is 0. Hence, $\underset{x \to 0}{L} U(x) \neq U(0)$, and $U(x)$ is discontinuous at $x = 0$.

Thus a merely convergent series of continuous functions does not necessarily represent a continuous function.

157. Integration of Real Series.—Let $\sum u_n(x)$ be a convergent series of real functions continuous in an interval (a, b). We say that the series can be integrated term by term between limits x_0 and x_1 lying within (a, b), if the sum-function $U(x)$ is integrable between these limits and the series

$$\sum \int_{x_0}^{x_1} u_n(x)dx$$

converges to the limit

$$\int_{x_0}^{x_1} U(x)dx.$$

Theorem.—*A series of continuous functions which converges uniformly in an interval (a, b) may be integrated term by term, provided the limits of integration are finite and lie in the interval (a, b):*

$$\int_{x_0}^{x_1} \left(\sum_1^{\infty} u_n(x)\right)dx = \sum_1^{\infty} \int_{x_0}^{x_1} u_n(x)dx.$$

For, by virtue of the uniform convergence of $\sum u_n(x)$, we may write

$$U_n(x) = U(x) - \overline{U}_n(x),$$

where $|\overline{U}_n(x)| < \epsilon$ for $n \geq N$ and for every x in (a, b), N being independent of x.

$$\int_{x_0}^{x_1} U_n(x)dx = \int_{x_0}^{x_1} U(x)dx - \int_{x_0}^{x_1} \overline{U}_n(x)dx,$$

or since

$$\int_{x_0}^{x_1} U_n(x)dx = \int_{x_0}^{x_1} \left(\sum_{i=1}^{n} u_i(x)\right) dx = \sum_{i=1}^{n} \int_{x_0}^{x_1} u_i(x)dx,$$

we have

$$\sum_1^{n} \int_{x_0}^{x_1} u_i(x)dx = \int_{x_0}^{x_1} U(x)dx - \int_{x_0}^{x_1} \overline{U}_n(x)dx.$$

Now

$$\left|\int_{x_0}^{x_1} \overline{U}_n(x)dx\right| \leqq \int_{x_0}^{x_1} |\overline{U}_n(x)||dx| < \epsilon\,|x_1 - x_0|,$$

hence

$$\underset{n}{L} \int_{x_0}^{x_1} \overline{U}_n(x)dx = 0,$$

and we get

$$\underset{n}{L} \sum_{i=1}^{n} \int_{x_0}^{x_1} u_n(x)dx = \int_{x_0}^{x_1} U(x)dx,$$

or

$$(a) \qquad \int_{x_0}^{x_1} \left(\sum_{1}^{\infty} u_n(x) \right) dx = \sum_{1}^{\infty} \int_{x_0}^{x_1} u_n(x)dx.$$

158. Theorem.—*If a uniformly convergent series be integrated term by term, the resulting series of integrals is uniformly convergent.*

Let $\sum_{1}^{\infty} u_n(x)$ be the given series, uniformly convergent in an interval (a, b), and let $U_{n,p}(x)$ be its partial remainder. Then since $\sum u_n(x)$ is uniformly convergent, we have

$$|U_{n,p}(x)| < \epsilon$$

for $n > N$, every p, and every x in (a, b), however small ϵ may be. The partial remainder in the series

$$\sum_{1}^{\infty} \int_{x_0}^{x_1} u_n(x)dx$$

is

$$\sum_{n+1}^{n+p} \int_{x_0}^{x_1} u_i(x)dx = \int_{x_0}^{x_1} \left(\sum_{n+1}^{n+p} u_i(x) \right) dx = \int_{x_0}^{x_1} U_{n,p}(x)dx.$$

But

$$\left|\int_{x_0}^{x_1} U_{n,p}(x)dx\right| \leqq \int_{x_0}^{x_1} |U_{n,p}\,(x)| \cdot |dx| < \epsilon\,|x_1 - x_0|$$

for $n > N$, every p, and every x in (a, b). Hence,

$$\sum_1^\infty \int_0^{x_1} u_n(x)dx$$

is uniformly convergent in the interval (a, b).

159. Differentiation of Real Series.—Let $\sum u_n(x)$ be a convergent series of real functions, whose terms possess definite derivatives of the first order at the point $x = x_0$. We say that the series can be differentiated term by term at the point $x = x_0$ when the sum-function $U(x)$ has a derivative at $x = x_0$ and the series $\sum u'_n(x)$ converges to the limit $U'(x_0)$, where the accent denotes differentiation: $u'_n(x) \equiv \frac{d}{dx}u_n(x)$, and $U'(x_0) \equiv \frac{d}{dx}U(x)\big|_{x=x_0}$.

Theorem.—*Any convergent series of functions may be differentiated term by term if the resulting series is uniformly convergent:*

If $\sum u_n(x)$ *is convergent with sum* $U(x)$, *then if the series* $\sum u'_n(x)$ *is uniformly convergent in an interval* (a, b), *its sum is* $U'(x)$:

$$(a) \qquad \frac{d}{dx}\left(\sum_1^\infty u_n(x)\right) = \sum_1^\infty u'_n(x) = \sum_1^\infty \frac{d}{dx} u_n(x),$$

provided the latter series is uniformly convergent in (a, b).

For, since $\sum u'_n(x)$ converges uniformly in (a, b), by Art. 157 we may integrate it term by term, obtaining the series

$$\sum_1^\infty \int_{x_0}^x u'_n(x)dx = \sum_1^\infty \left\{ u_n(x) - u_n(x_0) \right\}$$

$$= \sum_1^\infty u_n(x) - \sum_1^\infty u_n(x_0) = U(x) - U(x_0).$$

The sum of this series must be $\int_{x_0}^{x} V(x)dx$, where $V(x)$ is the sum of $\sum u'_n(x)$. Hence,

$$\int_{x_0}^{x} V(x)dx = U(x) - U(x_0),$$

which shows that $V(x)$ is the derivative of $U(x)$.

160. Quasi-uniform Convergence.—Uniform convergence is a *sufficient* but *not* a *necessary* condition for continuity of the sum-function, as can be shown by examples. Arzelà has given the necessary and sufficient condition that the sum of a series of continuous functions shall be continuous in terms of what he calls "convergenza uniforme a tratti" (generally called *quasi-uniform convergence* in English).

A series is said to be quasi-uniformly convergent when it converges for every value of x in the interval considered, and when to every positive ε, however small, and to any given positive integer n, however great, there corresponds a finite positive integer n′ > n, such that for every value of x in the given interval there exists an integer n_x comprised between n and n′ for which

$$|\overline{U}_{n_x}(x)| < \epsilon.$$

161. Uniform Convergence of a Series of Functions of Several Variables.—The definition of uniform convergence may be readily extended to series in several variables. Let $\sum_{1}^{\infty} u_n(x, y)$ be a series whose terms are functions of two independent variables x and y, and let us suppose that this series converges for all points (x, y) lying in a region R bounded by a closed contour C. *This series is* **uniformly convergent in the region R,** *or* **with respect to x and y,** *if, for any* *arbitrarily small* $\epsilon > o$, *an integer* N *can be found such that*

$$|\overline{U}_n(x, y)| < \epsilon$$

for $n > N$ *and for every point* (x, y) *in the region* R.

It can be shown that *the sum of such a series is a continuous function of the two variables* x *and* y *in this region, provided the terms of the series are all continuous in this region* R.

The theorem on term-by-term integration may also be generalized. *If each of the terms of the series is continuous in* R, *and if* $U(x, y)$ *is the sum of the series, we have*

$$\int\int U(x, y)dxdy = \sum_1^\infty \int\int u_n(x, y)dxdy,$$

where each of the double integrals is extended over the interior of any region inside R, *provided the given series is uniformly convergent in* R.

162. Uniform Convergence of a Sequence of Functions.—Let

$$f_1(x), f_2(x), f_3(x), \quad . \quad . \quad . \quad , f_n(x), \quad . \quad . \quad .$$

be a sequence whose elements are functions of a variable x (real or complex). Suppose this sequence converges for every x in a region R, and denote the limit by $F(x)$.

If for any $\epsilon > o$, *as small as we please, a positive integer* N, *independent of* x, *can be determined such that for every* $n \geqq N$, *we have*

$$|F(x) - f_n(x)| < \epsilon$$

for every x *in* R, *we say that the sequence is* **uniformly convergent** *and that* $f_n(x)$ *converges uniformly to its limit* $F(x)$ *in the region* R.

By a slight modification of the proof of Art. **155,** we can show that: *If the elements* $f_n(x)$ *of a uniformly convergent sequence are continuous functions of* x *for every* n *in a region* R, *the limit* $F(x)$ *is itself a continuous function of* x *in* R.

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CHAPTER XIII

POWER SERIES

163. Power Series.—A series of the form

$$P = a_0 + a_1z + a_2z^2 + a_3z^3 + \ . \ . \ . \ = \sum_{n=0}^{\infty} a_nz^n,$$

where the a_i are real or complex constants and z is a real or complex variable, is called a **power series.**

Since P reduces to a_0 when $z = 0$, we see that every power series converges for *at least one point.* On the other hand, there are power series which converge at but one point, as for example,

$$a_0 + 1!x + 2!x^2 + 3!x^3 + \ . \ . \ . \ ,$$

which diverges for $x \neq 0$, since $\underset{n}{L} n!x^n = \infty$.

Most power series converge for some values of z and diverge for others.

A more general form of power series is

$$P = a_0 + a_1(z - a) + a_2(z - a)^2 + a_3(z - a)^3 + \ . \ . \ .$$

We say P is *developed about the point a.*

164. Theorem.—*If when $z = z_0$ each term of the power series $\sum a_nz^n$ is in absolute value < some finite positive constant c: $|a_nz_0{}^n| < c$ for every n, then the power series is absolutely convergent for every $|z| < |z_0|$.*

For, from $|a_nz_0{}^n| < c$, we have

$$\left|a_nz^n\right| = \left|a_nz_0{}^n\right| \cdot \left|\frac{z}{z_0}\right|^n < c\left|\frac{z}{z_0}\right|^n$$

for every n. Hence, each term of $\sum |a_n z^n|$ is < the corresponding term of $\sum c \left|\frac{z}{z_0}\right|^n$; but this latter series is a geometric series which converges for $\left|\frac{z}{z_0}\right| < 1$, or $|z| < |z_0|$. Therefore, by Art. **92,** the series $\sum |a_n z^n|$ converges and hence $\sum a_n z^n$ converges absolutely for $|z| < |z_0|$.

165. Theorem.—*If* $\sum a_n z^n$ *converges for* $z = z_0$, *it converges absolutely for* $|z| < |z_0|$.

For if $\sum a_n z_0^n$ is convergent, all its terms have finite values, and the conditions of Art. **164** are satisfied.

166. Theorem.—*If* $\sum a_n z^n$ *is non-convergent for* $z = z_0$, *it is also non-convergent for every* $|z| > |z_0|$.

For if $\sum a_n z^n$ were convergent for a value of z such that $|z| > |z_0|$, it would also converge for $z = z_0$, by Art. **165.**

167. Circle of Convergence.—If we put in a class A_1 all positive values of z for which $\sum a_n z^n$ converges, and in a class A_2 all those for which it is non-convergent, then by Arts. **165, 166,** it follows that every number of A_1 will be < every number of A_2. This separation includes all the positive real numbers, hence the cut (A_1, A_2) defines a real positive number ρ, which has the property that for every $|z| < \rho$, the power series $\sum a_n z^n$ is absolutely convergent, and for every $|z| > \rho$, the series is non-convergent. Hence, $\sum a_n z^n$ converges absolutely in the circle with radius ρ and center at the origin, and is non-convergent at all points outside this circle. This circle is called the **circle of convergence** of the power series $\sum a_n z^n$ and ρ is called the **radius of convergence.**

The preceding demonstration gives us no information as to what happens on the circumference of the circle of convergence, for $|z| = \rho$. At these points the series may be absolutely convergent, simply convergent, or non-convergent.

For a real power series, the region of convergence becomes an **interval of convergence,** and ρ is called in this case the **limit of convergence.**

Examples.—(1) The power series

$$1 + x + x^2 + x^3 + \dots$$

has the interval of convergence $-1 < x < 1$; both end-points of the interval, $x = \pm 1$, must be excluded.

(2) The series

$$1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

has the interval of convergence $-1 \leqq x < 1$; the upper end-point $x = 1$ is to be excluded.

(3) The series

$$1 + \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \dots$$

has the interval of convergence $-1 \leqq x \leqq 1$; both end-points $x = \pm 1$ being included.

(4) The series

$$1 + 1!x + 2!x^2 + 3!x^3 + \dots$$

is convergent only for $x = 0$, and non-convergent for all other points.

(5) The series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

is convergent for all finite values of x.

168. Determination of the Radius of Convergence.—By the ratio test of Art. **101,** the power series $\sum |a_n z^n|$ converges if

$$\underset{n}{L} \left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right| < 1,$$

that is, if

$$|z| < \underset{n}{L}\left|\frac{a_n}{a_{n+1}}\right|.$$

Similarly, $\sum |a_n z^n|$ is divergent if $|z| > \underset{n}{L}\left|\frac{a_n}{a_{n+1}}\right|$. Hence, if $\underset{n}{L}\left|\frac{a_n}{a_{n+1}}\right|$ exists, $=\rho$, the power series converges absolutely for $|z| < \rho$, and is non-convergent for $|z| > \rho$, and we have the

Theorem.—*If* $L\left|\frac{a_n}{a_{n+1}}\right|$ *exists,* $=\rho$, *then* ρ *is the radius of convergence of* $\sum a_n z^n$.

Applying similarly the radical test of Art. **103,** we obtain the

Theorem.—*If* $L\sqrt[n]{|a_n|}$ *exists* $= r$, *then* $\frac{1}{r}$ *is the radius of convergence of* $\sum a_n z^n$.

169. Uniform Convergence of a Power Series. Theorem. *The power series* $\sum a_n z^n$ *is uniformly convergent at all points within the circle of radius* ρ', *where* $\rho' < \rho$, *and* ρ *is the radius of convergence of* $\sum a_n z^n$.

For, if $\rho' < \rho$, the series converges absolutely for $z = \rho'$, and the remainder for that value can be made arbitrarily small:

$$\{|a_{n+1}\rho'^{n+1}| + \ldots + |a_{n+p}\rho'^{n+p}| + \ldots\} < \epsilon,\ n \geqq N.$$

But if $|z| < \rho'$ we have

$$|a_{n+1}z^{n+1} + \ldots + a_{n+p}z^{n+p} + \ldots| < \{|a_{n+1}\rho'^{n+1}| + \ldots + |a_{n+p}\rho'^{n+p}| + \ldots\} < \epsilon,\ n \geqq N,$$

hence $\sum a_n z^n$ is uniformly convergent for every $|z| < \rho'$.

170. Continuity of a Power Series. Theorem.—*The sum* $f(z)$ *of a power series* $\sum a_n z^n$ *is a continuous function of* z *for all points within the circle of convergence.*

For, if z_0 is any number in absolute value $< \rho$, the radius of convergence, it is evident that a number ρ' may be found which is $< \rho$ and $> |z_0|$. Then by Art. **169**, $\sum a_n z^n$ converges uniformly in the circle of radius ρ', and hence by Art. **155** the sum-function $f(z)$ is continuous at the point z_0, since z_0 belongs to the region of uniform convergence.

This proof does not apply to points on the circumference of the circle of convergence. The question of what happens there is answered by Abel's theorem in the following article.

171. Abel's Theorem on the Continuity of a Power Series. *If a real power series* $\sum a_n x^n$ *converges at an end-point of its interval of convergence, the domain of uniform convergence extends up to and includes that point, and the continuity of its sum-function* $f(x)$ *extends up to and includes that point:*

$$\underset{x\to\rho}{L} \sum_0^\infty a_n x^n = \sum_0^\infty a_n \rho^n,$$

provided $\sum\limits_0^\infty a_n \rho^n$ *converges.*

If the power series converges at the point $x = \rho$, then N can be found such that

$$|a_{n+1}\rho^{n+1} + \ . \ . \ . + a_{n+p}\rho^{n+p}| < \epsilon,\ n \geq N,$$
$$p = 1, 2, 3, \ . \ . \ .$$

Now we may write

$$\begin{aligned} &a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \ . \ . \ . + a_{n+p}x^{n+p} \\ &= a_{n+1}\rho^{n+1}\left(\frac{x}{\rho}\right)^{n+1} + a_{n+2}\rho^{n+2}\left(\frac{x}{\rho}\right)^{n+2} + \ . \ . \ . \\ &\qquad + a_{n+p}\rho^{n+p}\left(\frac{x}{\rho}\right)^{n+p}, \end{aligned}$$

and if we take $0 < x < \rho$, all the factors $\left(\frac{x}{\rho}\right)^{n+1}, \ . \ . \ . \ ,$ $\left(\frac{x}{\rho}\right)^{n+p}$ are positive and decreasing. Put $\frac{x}{\rho} = c$, then if

$$R_{n,p}(x) = a_{n+1}x^{n+1} + \ . \ . \ . + a_{n+p}x^{n+p},$$

we have

$$R_{n,p}(x) = R_{n,1}(\rho) \cdot c^{n+1} + [R_{n,2}(\rho) - R_{n,1}(\rho)]c^{n+2} + \\ \quad . \; . \; . + [R_{n,p}(\rho) - R_{n,p-1}(\rho)]c^{n+p} \\ = R_{n,1}(\rho)[c^{n+1} - c^{n+2}] + R_{n,2}(\rho)[c^{n+2} - c^{n+3}] + \; . \; . \; . \\ + R_{n,p-1}(\rho)[c^{n+p-1} - c^{n+p}] + R_{n,p}(\rho) \cdot c^{n+p}.$$

Now $(c^{n+1} - c^{n+2})$, . . . , $(c^{n+p-1} - c^{n+p})$, c^{n+p} are all positive, and $|R_{n,1}(\rho)|$, ... , $|R_{n,p}(\rho)|$ are all $< \epsilon$ for $n \geqq N$, $p = 1, 2, 3,$ . . . . . Hence

$$|R_{n,p}(x)| < \epsilon\,[(c^{n+1} - c^{n+2}) + (c^{n+2} - c^{n+3}) + \; . \; . \; . \\ + c^{n+p}] < \epsilon \cdot c^{n+1} < \epsilon$$

for $n \geqq N$, and for every p, since $c = \frac{x}{\rho}$, and $0 < x < \rho$, therefore $c < 1$. This inequality $|R_{n,p}(x)| < \epsilon$ also holds for all values of x in the interval $o < x \leqq \rho$, since $|R_{n,p}(\rho)| < \epsilon$. Therefore, $0 < x \leqq \rho$ is an interval of uniform convergence, and the uniformity of convergence extends up to and includes the end-point ρ of the interval of convergence.

It follows from the uniformity of the convergence up to the limit ρ that the sum-function $f(x)$ is a continuous function of x in the interval $0 < x \leqq \rho$.

Hence,

$$\underset{x \to \rho}{L} f(x) = f(\rho).$$

172. Abel's Theorem on the Multiplication of Series.— The theorem of Art. **147** may be proved very easily by use of the properties of power series and the theorem of Abel just established. Using the notation of Art. **147,** each of the three power series

$$f(x) = \sum u_n x^n,\ \phi(x) = \sum v_n x^n,\ \psi(x) = \sum w_n x^n$$

converges, by hypothesis, for $x = 1$, and hence each of them converges absolutely for any value of x between -1 and $+1$. For any such value of x, Cauchy's theorem on the multiplication of absolutely convergent series (Art. **146**) applies, and we get

$$f(x).\ \phi(x) = \psi(x).$$

By Abel's theorem of Art. **171,** as $x \to 1$ the three functions $f(x)$, $\phi(x)$, $\psi(x)$ approach limits U, V, W respectively. Since the two sides of the equation

$$f(x).\ \phi(x) = \psi(x)$$

remain equal, we shall have in the limit $W = UV$.

173. Differentiation of Power Series.—Let $(-\rho, \rho)$ be the interval of convergence of the real power series $\sum a_n x^n$. By Art. **159,** we may differentiate this series term by term provided the resulting series is uniformly convergent. The resulting power series is

$$a_1 + 2a_2x + 3a_3x^2 + \ . \ . \ . + na_nx^{n-1} + \ . \ . \ . \\ = \sum_0^\infty (n+1)a_{n+1}x^n.$$

This series converges uniformly within any interval contained within its interval of convergence. We shall show that its interval of convergence is the same as that of the original power series $\sum a_n x^n$, *viz.* $(-\rho, \rho)$.

Put $|a_n| = \alpha_n$, $|x| = X$, then our proposition will be proved if we show that the series

$$\alpha_1 + 2\alpha_2X + 3\alpha_3X^2 + \ . \ . \ . + n\alpha_nX^{n-1} + \ . \ . \ .$$

is convergent for $X < \rho$ and divergent for $X > \rho$.

First suppose $X < \rho$. Then if we take ρ' between X and ρ, the series

$$\frac{1}{\rho'} + \frac{2}{\rho'}\left(\frac{X}{\rho'}\right) + \frac{3}{\rho'}\left(\frac{X}{\rho'}\right)^2 + \ . \ . \ . + \frac{n}{\rho'}\left(\frac{X}{\rho'}\right)^{n-1} + \ . \ . \ .$$

is convergent, since the ratio of any term to the preceding approaches the limit $\frac{X}{\rho'}$, which is < 1. If we multiply the terms of this series by the factors

$$\alpha_1\rho', \alpha_2\rho'^2, \alpha_3\rho'^3, \quad . \quad . \quad . \quad , \alpha_n\rho'^n, \quad . \quad . \quad .$$

applying Art. **125,** we obtain a new series

$$\alpha_1 + 2\alpha_2X + 3\alpha_3X^2 + \quad . \quad . \quad . \quad + n\alpha_nX^{n-1} + \quad . \quad . \quad . \quad ,$$

which must converge. Hence, the differentiated series converges for $X < \rho$.

Now suppose $X > \rho$. If we should assume that

$$\alpha_1 + 2\alpha_2X + 3\alpha_3X^2 + \quad . \quad . \quad . \quad + n\alpha_nX^{n-1} + \quad . \quad . \quad .$$

were convergent, then the series

$$\alpha_1X + 2\alpha_2X^2 + 3\alpha_3X^3 + \quad . \quad . \quad . \quad + n\alpha_nX^n + \quad . \quad . \quad .$$

would also be convergent, and consequently the series $\sum_1^\infty \alpha_nX^n$ would also converge, since each of its terms is $<$ the corresponding term of the preceding series. Then $\sum a_nx^n$ would converge outside its interval of convergence, hence $\sum n\alpha_nX^{n-1}$ cannot converge for $X > \rho$, and must therefore be divergent for every $X > \rho$.

Hence, *the series resulting from the differentiation of* $\sum a_nx^n$ *term by term has the same interval of convergence as the original series.*

Let us denote the sum of the series $\sum na_nx^{n-1}$ by $f_1(x)$; then $f_1(x)$ is a continuous function of x within the interval $(-\rho, \rho)$. By Art. **159,** since this series converges uniformly in any interval $(-\rho', \rho')$, where $\rho' < \rho$, $f_1(x)$ is the derivative of $f(x)$ throughout such an interval. Since ρ' may be chosen as near ρ as we please, we may say that $f(x)$ has a derivative for any value of x between $-\rho$ and ρ, and that this derivative is represented by the series

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots,$$

obtained by differentiating the given series term by term.

Repeating the above reasoning on the series $\sum na_nx^{n-1} = f'(x)$, we see that $f(x)$ has a second derivative:

$$f''(x) = 2a_2 + 6a_3x + \dots + n(n-1)a_nx^{n-2} + \dots,$$

and so on, so that $f(x)$ has an unlimited set of successive derivatives in the interval $(-\rho, \rho)$, and

$$f^{(n)}(x) = n!a_n + 2 \cdot 3 \dots n(n+1)a_{n+1}x + \dots$$

Although the derived series has the same interval of convergence as the original series, a power series may converge at an extremity of the interval, and the derived series may diverge there.

For example, the series

$$x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \dots + \frac{x^n}{n^2} + \dots$$

converges for $x = 1$, and the derived series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \dots + \frac{x^{n-1}}{n} + \dots$$

diverges for $x = 1$. The interval of convergence for both is $-1 < x < 1$.

174. Maclaurin's Theorem.—If in the formulæ of Art. **173** we put $x = 0$, we find

$$a_0 = f(0), a_1 = f'(0), a_2 = \frac{f''(0)}{2!}, \dots, a_n = \frac{f^{(n)}(0)}{n!}, \dots$$

Hence,

$$(a) \quad f(x) = \sum_0^\infty a_nx^n = f(0) + \frac{x}{1} \cdot f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots,$$

which is **Maclaurin's development.**

Examples.—(1) If we take $f(x) = \sin x$, we have

$$\begin{aligned} f'(x) &= \cos x, & f'(0) &= 1, \\ f''(x) &= -\sin x, & f''(0) &= 0, \\ f'''(x) &= -\cos x, & f'''(0) &= -1, \\ f^{(\text{IV})}(x) &= \sin x, & f^{(\text{IV})}(0) &= 0, \\ \cdots & & \cdots & \end{aligned}$$

Hence, we find

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{b}$$

Similarly we obtain the expansion

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{c}$$

(2) If $f(x) = \log(1 + x)$, we get

$$\begin{aligned} f'(x) &= \frac{1}{1+x}, & f'(0) &= 1, \\ f''(x) &= -\frac{1}{(1+x)^2}, & f''(0) &= -1, \\ f'''(x) &= \frac{2}{(1+x)^3}, & f'''(0) &= 2, \\ f^{(\text{IV})}(x) &= -\frac{2.3}{(1+x)^4}, & f^{(\text{IV})}(0) &= -2\cdot 3, \\ \cdots & & \cdots & \end{aligned}$$

Hence,

$$\log(1 + x) = x - \frac{x^2}{2} + 2\cdot\frac{x^3}{3!} - 2\cdot 3\cdot\frac{x^4}{4!} + \cdots,$$

or

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \tag{d}$$

175. Integration of Power Series.—By Art. **157** we may integrate a power series $\sum a_nx^n$ term by term, obtaining a new power series which has an arbitrary constant term and which converges in the same interval of convergence as the given series:

$$\int f(x)dx = c_0 + a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + \cdots + a_n\frac{x^{n+1}}{n+1} + \cdots$$

176. Theorem.—*If a power series* $\sum u_nx^n$ *is zero for every value of* x *in some interval* $(-\alpha, \alpha)$, *every coefficient* u_n *must be zero.*

For, if we put $x = 0$ in the given series, the series reduces to u_0, and therefore $u_0 = 0$. We can then write

$$x(u_1 + u_2x + u_3x^2 + \cdots) = 0.$$

From this equation it follows that

$$u_1 + u_2x + u_3x^2 + \cdots = 0,$$

provided $x \neq 0$, but it does not follow that this last equation is satisfied when $x = 0$, and therefore u_1 cannot be shown to vanish by putting $x = 0$ here as in the previous case. But by applying Abel's theorem on the continuity of a power series (Art. **171**), we can avoid this difficulty. Let

$$f_1(x) = u_1 + u_2x + u_3x^2 + \cdots,$$

then since $f_1(x)$ is, by Abel's theorem, a continuous function of x, we have

$$\underset{x\to 0}{L}\, f_1(x) = f_1(0) = u_1.$$

But $\underset{x\to 0}{L}\, f_1(x) = 0$, therefore $u_1 = 0$. By repeating this reasoning, each of the following coefficients can be shown to be zero.

177. Theorem.—*If the two power series* $\sum a_n x^n$ *and* $\sum b_n x^n$ *have the same sum for all values of* x *in the neighborhood of* $x = 0$, *the coefficients of the same powers of* x *in the two series must be identical.*

For, if we write

$$\sum a_n x^n - \sum b_n x^n = 0,$$

we have

$$\sum (a_n - b_n)x^n = 0,$$

and by Art. **176**, each coefficient $a_n - b_n$ must be 0, therefore $a_n = b_n$.

178. Method of Undetermined Coefficients.—The theorem of Art. **177** is the basis of the method of undetermined coefficients. It rests on the hypothesis that a given function $f(x)$ can be developed in a power series in some interval about a point, say the origin. We put

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ . \ . \ . \ ,$$

where the a's are undetermined coefficients, and then we seek enough relations between the a's to determine as many of them as we need.

Example.—Let us seek to determine the function $f(z) = \sum a_n z^n$ which satisfies the functional equation

$$f(z) \cdot f(w) = f(z + w)$$

for all values of z and w.

Substituting in the functional equation, we get

$$(a_0 + a_1 z + a_2 z^2 + \ . \ . \ . \ )(a_0 + a_1 w + a_2 w^2 + \ . \ . \ . \ ) = a_0 + a_1(z + w) + a_2 (z + w)^2 + \ . \ . \ . \ ,$$

or expanding and arranging the terms in powers of z and w,

$$\begin{aligned}
&a_0^2 + a_1a_0z + a_0a_1w + a_2a_0z^2 + a_1^2zw + a_0a_2w^2 + \dots \\
&+ a_na_0z^n + a_{n-1}a_1z^{n-1}w + \dots + a_{n-k}a_kz^{n-k}w^k + \dots \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + a_0a_nw^n \\
&+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\quad = a_0 + a_1z + a_1w + a_2z^2 + 2a_2zw + a_2w^2 + \dots \\
&+ a_nz^n + na_nz^{n-1}w + \dots + n_ka_nz^{n-k}w^k + \dots \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + a_nw^n,
\end{aligned}$$

where

$$n_k = \frac{n(n-1)\ \dots\ (n-k+1)}{k!}.$$

Equating coefficients of like powers of z and w on both sides of this equation, we get

$$a_{n-k}\,a_k = a_n \cdot n_k.$$

$$\begin{gathered}
\therefore a_0^2 = a_0,\ a_0 = 1,\ a_1^2 = 2a_2,\ a_2a_1 = 3a_3, \\
a_3a_1 = 4a_4,\ \dots,\ a_{n-1}a_1 = na_n.
\end{gathered}$$

Multiply these equations together and we get

$$a_1^n = a_n \cdot n!, \text{ or } a_n = \frac{a_1^n}{n!}.$$

a_1 is an undetermined constant, let us call it simply a. Then any series of the form

$$f(z) = \sum_0^\infty a_nz^n, \quad \text{where } a_n = \frac{a^n}{n!}$$

will satisfy the functional equation.

179. Laurent Series.—*A* **Laurent series** *is a series of positive and negative powers of a variable; or a two-way power series;* it may be represented by

$$\sum_{-\infty}^{+\infty} a_nz^n, \text{ or } \sum_0^\infty a_nz^n + \sum_1^\infty a_{-n}z^{-n}.$$

A series of negative powers of z, as $\sum_{1}^{\infty} a_{-n}z^{-n}$, which converges for a value z_0 of z, is absolutely convergent for every z such that $\left|\frac{1}{z}\right| < |z_0|$ or $|z| > |z_0|$. The region of convergence of such a series will then be represented by the whole plane with the exception of the area of a circle whose center is at the origin; *i.e.*, the exterior of a circle.

If a series of positive and negative powers of z, as $\sum_{-\infty}^{+\infty} a_n z^n$, converges for $z = z_0$, then the series $\sum_{0}^{\infty} a_n z^n$ is absolutely convergent for every $|z| < |z_0|$, and the series $\sum_{1}^{\infty} a_{-n}z^{-n}$ is absolutely convergent for every $|z| > |z_0|$. Hence, *the region of convergence of a Laurent series will in general be formed by a circular ring contained between the circumferences of two circles with centers at the origin;* in special cases it may be the whole plane, or only the points of a circumference.

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CHAPTER XIV

ELEMENTARY FUNCTIONS

180. Representation of the Number e by a Series.—If we expand $\left(1 + \frac{1}{n}\right)^n$ by the binomial theorem, for n a positive integer, we get

$$\left(1+\frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}\frac{1}{n^r} + \cdots,$$

which may be written

$$(a)\quad \left(1+\frac{1}{n}\right)^n = 1 + \frac{1}{1!} + \frac{1\left(1-\frac{1}{n}\right)}{2!} + \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!} + \cdots + \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)}{r!} + \cdots$$

We may apply Tannery's theorem (Art. **154**) to this, taking the convergent series $\sum \frac{1}{r!}$ for $\sum M_r$. Then taking the limits of the terms, we get

$$(b)\quad L\left(1+\frac{1}{n}\right)^n = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{r!} + \cdots$$

This series is called the **e series.** It is used for the numerical calculation of the number e. The computation may be arranged as follows:

$$
\begin{array}{l}
2. \\
0.5 \\
0.1666666667\ \ldots \\
0.0416666667\ \ldots \\
0.0083333333\ \ldots \\
0.0013888889\ \ldots \\
0.0001984127\ \ldots \\
0.0000248016\ \ldots \\
0.0000027557\ \ldots \\
0.0000002756\ \ldots \\
0.0000000251\ \ldots \\
\hline
\end{array}
$$

$$e = 2.7182818\ \ldots$$

The value of e to 25 decimal places is

$$e = 2.7\ 1828\ 1828\ 45\ 90\ 45\ 23536\ 02874\ \ldots$$

Its value to 205 decimal places has been computed by W. Shanks.

181. The Exponential Theorem.—For real values of x we shall assume that the function e^x has been defined as an exponential and its properties known.

The power series

$$\sum_{0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

is called the **exponential series.** By the ratio test we find that it converges for every finite value of x. Let us denote its sum by $E(x)$:

$$E(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

and study its properties for real values of x.

Differentiating (Art. **173**), we find

$$E'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

so that for any real x,

$$E'(x) = E(x).$$

Now take the function

$$f(x) = \frac{E(x)}{e^x},$$

and differentiating, we find

$$f'(x) = \frac{e^x E'(x) - E(x)e^x}{e^{2x}} = \frac{E'(x) - E(x)}{e^x} = 0,$$

since $E'(x) = E(x)$. From $f'(x) = 0$, we get $f(x) = C$, a constant; putting $x = 0$ we have $f(0) = 1$, therefore $C = 1$. Hence, $E(x) = e^x$, and we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdot \cdot \cdot$$

for any real x. This result is known as the **exponential theorem.**

Another method of studying the properties of the function $E(x)$, and of deriving the exponential theorem is the following. Taking

$$E(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdot \cdot \cdot + \frac{x^n}{n!} + \cdot \cdot \cdot ,$$

$$E(y) = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdot \cdot \cdot + \frac{y^n}{n!} + \cdot \cdot \cdot ,$$

we can apply the rule for multiplication of these two series (Art. **146**) since they are both absolutely convergent for all real finite values of x and y, and we thus obtain

$$\begin{aligned} E(x) \cdot E(y) &= 1 + (x + y) + \left(\frac{x^2}{2!} + xy + \frac{y^2}{2!}\right) + \\ &\quad + \left(\frac{x^3}{3!} + \frac{x^2}{2!} \cdot \frac{y}{1} + \frac{x}{1} \cdot \frac{y^2}{2!} + \frac{y^3}{3!}\right) + \cdot \cdot \cdot \\ &= 1 + (x + y) + \frac{(x + y)^2}{2!} + \frac{(x + y)^3}{3!} + \cdot \cdot \cdot \\ &= E(x + y). \end{aligned}$$

By repeated application of this result, we get

$$E(x) \cdot E(y) \cdot E(z) = E(x + y) \cdot E(z) = E(x + y + z),$$

and so on for any number of factors.

If we put $x = 0$ and $x = 1$ in $E(x) = \sum_0^\infty \frac{x^n}{n!}$, we find $E(0) = 1$, $E(1) = e$.

Now let x be a positive integer m, then

$$\begin{aligned} E(m) &= E(1 + 1 + 1 + \ . \ . \ . \text{ to } m \text{ terms}) \\ &= E(1) \cdot E(1) \cdot E(1) \ . \ . \ . \text{ to } m \text{ factors} \\ &= [E(1)]^m = e^m. \end{aligned}$$

If x is a positive fraction $\frac{p}{q}$, we have

$$\begin{aligned} \left[E\left(\frac{p}{q}\right)\right]^q &= E\left(\frac{p}{q}\right) \cdot E\left(\frac{p}{q}\right) \ . \ . \ . \text{ to } q \text{ factors} \\ &= E\left(\frac{p}{q} + \frac{p}{q} + \ . \ . \ . \text{ to } q \text{ terms}\right) \\ &= E(p) = e^p, \end{aligned}$$

since p is an integer.

$$\therefore E\left(\frac{p}{q}\right) = e^{\frac{p}{q}}.$$

If x is a negative rational number $-s$, we have

$$E(-s) \cdot E(s) = E(-s + s) = E(0) = 1,$$

$$\therefore E(-s) = \frac{1}{E(s)} = \frac{1}{e^s} = e^{-s}.$$

Hence, when x is rational, we have

$$E(x) = e^x.$$

This relation could also be proved for irrational values of x by regarding this irrational value as defined by a sequence of rational numbers.

Still another derivation of the exponential theorem is as follows. We recall first that

$$e^x = \underset{n}{L}\left(1 + \frac{x}{n}\right)^n.$$

Expanding $\left(1 + \frac{x}{n}\right)^n$ by the binomial theorem (for n a positive integer), we have

$$\left(1 + \frac{x}{n}\right)^n = 1 + n \cdot \frac{x}{n} + \frac{n(n-1)}{2!}\frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{x^3}{n^3} + \cdot \cdot \cdot ,$$

or

$$(b)\quad \left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{1 - \frac{1}{n}}{2!}x^2 + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{3!}x^3 + \cdot \cdot \cdot$$

Regarding this series as a series of functions of n, and applying Tannery's theorem (Art. **154**) by comparing the terms of this series with the corresponding terms of the series

$$1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} + \cdot \cdot \cdot ,$$

which is convergent for every x, and whose terms are constant as regards n, we may take the limits of the terms of (b) as $n \to \infty$, and we obtain

$$e^x = \underset{n}{L}\left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdot \cdot \cdot .$$

182. The Exponential Series in a Complex Variable.—Let us consider now the series

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdot \cdot \cdot + \frac{z^n}{n!} + \cdot \cdot \cdot ,$$

where z is complex, $= x + iy$. This series converges over all the finite part of the z-plane. Denote its sum by $E(z)$.

By multiplication of series, as in Art. **181,** we find

$$(a) \qquad E(z_1) \cdot E(z_2) = E(z_1 + z_2),$$

and from this, by the same methods as in Art. **181,** we show that for z any real number,

$$E(z) = e^z.$$

We have, if θ is real,

$$\begin{aligned} E\,(i\theta) &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + i\sin\theta, \end{aligned}$$

using the expansions of $\cos\theta$ and $\sin\theta$ in Art. **174.** This is often written

$$(b) \qquad e^{i\theta} = \cos\theta + i\sin\theta.$$

This is **Euler's theorem.**

Since

$$\begin{aligned} E(i\theta) &= \cos\theta + i\sin\theta, \\ E(-i\theta) &= \cos\theta - i\sin\theta, \end{aligned}$$

we obtain by adding,

$$(c) \qquad \begin{cases} \cos\theta = \dfrac{1}{2}[E(i\theta) + E(-i\theta)] \\ \sin\theta = \dfrac{1}{2i}[E(i\theta) - E(-i\theta)]\cdot \end{cases}$$

By (a), we have

$$(d) \qquad E(x + iy) = E(x) \cdot E(iy) = e^x(\cos y + i\sin y).$$

We know from Trigonometry that $\cos y$ and $\sin y$ are unaltered by adding $2k\pi$ to y, where k is any positive or negative integer, hence we get from (d),

$$E(z) = E(z + 2k\pi i),$$

which shows that $E(z)$ *is a* **periodic** *function of* z *with* **period 2πi.**

183. Logarithm of a Complex Variable.—We have seen in the preceding Article that if x is real,

$$E(ix) = \cos x + i \sin x;$$

it follows from this that

$$E(2n\pi i) = \cos 2n\pi + i \sin 2n\pi = 1.$$

Since $E(x + iy) = e^x(\cos y + i \sin y)$, there can be no solutions of the equation $E(x + iy) = 1$ other than $x = 0$, $y = 2n\pi$.

It follows that if we wish to solve the equation $E(u) = v$, so as to obtain the function inverse to the exponential function, the value obtained will not be single-valued, but will be of the form $u = u_0 + 2n\pi i$ (n an integer), where u_0 is one solution.

If z is put in the form

$$z = r(\cos \theta + i \sin \theta) = re^{i\theta},$$

we have

(a) $$\log z = \log r + i\theta + 2n\pi i.$$

184. The Circular Functions of a Complex Variable.—The circular or trigonometric functions have so far been considered as defined by their usual geometric definitions as ratios, but obviously these definitions fail when we attempt to apply them to complex values of the argument. It is possible, however, to give purely analytic definitions of these functions, and to derive their properties from these new definitions, and so place the theory of the circular functions on a basis independent of all geometric considerations.

We have already seen that for real values of the variable, the sine and cosine functions are represented by the expansions

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

which are convergent for all finite values of x. It is quite natural then to define these functions for complex values of the variable by these same series, which converge absolutely over all the finite part of the complex plane.

For any complex z, then, *we define:*

$$(a) \qquad \begin{aligned} \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots, \\ \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \end{aligned}$$

To define the remaining trigonometric functions, we use the equations

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z},$$
$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.$$

It follows at once that

$$\cos z + i \sin z = E(iz).$$

From these analytic definitions, we can now deduce all the fundamental properties of the circular functions. By multiplication of series, we obtain without any difficulty the addition theorems

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2,$$
$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

It is evident from the series that

$$\cos(-z) = \cos z, \ \sin(-z) = -\sin z,$$

hence

$$\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2,$$

and putting $z_1 = z_2$, we find

$$\cos 0 = 1 = \cos^2 z + \sin^2 z.$$

It is not difficult to proceed in this manner and derive the other properties of the functions.

Another form of the definitions is the following. Since $\cos z + i \sin z = E(iz)$, we get $\cos z - i \sin z = E(-iz)$, and then by addition and subtraction,

$$(b) \qquad \begin{aligned} \cos z &= \tfrac{1}{2}[E(iz) + E(-iz)], \\ \sin z &= \frac{1}{2i}[E(iz) - E(-iz)]. \end{aligned}$$

Instead of taking the series (*a*) as definitions, we could have taken equations (*b*) as definitions in terms of the exponential function, and from the properties of the function $E(z)$ derive those of $\cos z$ and $\sin z$.

As illustrative of this method, we have

$$\cos z + i \sin z = E(iz), \quad \cos z - i \sin z = E(-iz),$$

multiplying, we get

$$\cos^2 z + \sin^2 z = E(iz) \cdot E(-iz) = E(0) = 1.$$

Again,

$$\begin{aligned} \cos(z_1 + z_2) &= \tfrac{1}{2}\{E(iz_1 + iz_2) + E(-iz_1 - iz_2)\} \\ &= \tfrac{1}{2}\{E(iz_1) \cdot E(iz_2) + E(-iz_1) \cdot E(-iz_2)\} \\ &= \tfrac{1}{4}\{E(iz_1) + E(-iz_1)\} \cdot \{E(iz_2) + E(-iz_2)\} \\ &\quad - \tfrac{1}{4}\{E(iz_1) - E(-iz_1)\} \cdot \{E(iz_2) - E(-iz_2)\} \\ &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2. \end{aligned}$$

The number π may be introduced analytically by defining it to be such that the numerically smallest root of $E(z) = 1$ is $2\pi i$.

185. Hyperbolic Functions.—*We may define the hyperbolic functions by the following equations:*

$$\cosh z = \tfrac{1}{2}[E(z) + E(-z)],$$
$$\sinh z = \tfrac{1}{2}[E(z) - E(-z)],$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{1}{\tanh z} = \frac{\cosh z}{\sinh z},$$
$$\operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z}.$$

The properties of these functions may be derived from those of the exponential function $E(z)$. These properties correspond closely to those of the trigonometric functions, as indicated by the similarity of the definitions.

The expansions of $\cosh z$ and $\sinh z$ in power series are:

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots,$$
$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

The relations between the hyperbolic and circular (trigonometric) functions are shown by the following identities, which follow immediately from the definitions:

$$\begin{aligned} \sinh z &= -i \sin iz, \\ \cosh z &= \cos iz, \\ \sin z &= -i \sinh iz, \\ \cos z &= \cosh iz. \end{aligned}$$

186. Series for Tan^{-1} x.—For any x between -1 and $+1$, we may write

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

Integrating this series term by term between the limits 0 and x, where $|x| < 1$, we find

$$\int_0^x \frac{dx}{1+x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots$$

But $\int_0^x \frac{dx}{1+x^2} = \tan^{-1} x$, hence

(a) $$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

For $x = 1$, we have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

This series converges very slowly, but by special devices we may obtain from this series much more rapidly converging series. One well-known method is to use the trigonometric identity

$$\frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239},$$

from which we have the formula

$$\frac{\pi}{4} = 4 \left\{ \frac{1}{5} - \frac{1}{3} \cdot \frac{1}{5^3} + \frac{1}{5} \cdot \frac{1}{5^5} - \dots \right\} - \left\{ \frac{1}{239} - \frac{1}{3} \cdot \frac{1}{239^3} + \dots \right\},$$

which may be used for the calculation of the value of π.

187. Series for Sin⁻¹ x.—We have by the binomial theorem,

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2} x^2 + \frac{1 \cdot 3}{2 \cdot 4} x^4 + \dots + \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n} x^{2n} + \dots,$$

for x between -1 and $+1$. Integrating term by term between 0 and x, where $|x| < 1$, we obtain the development

$$\sin^{-1} x = \int_0^x \frac{dx}{\sqrt{1 - x^2}} = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \cdots + \frac{1\cdot 3\cdot 5\ \ldots\ 2n-1}{2\cdot 4\cdot 6\ \ldots\ 2n}\cdot\frac{x^{2n+1}}{2n+1} + \cdots$$

188. The Logarithmic Series.—To derive the expansion of the logarithm in a power series, we shall assume the representation of the natural logarithm by a definite integral:

$$\int_0^x \frac{dx}{1 + x} = \log_e (1 + x)$$

as known. Starting with the geometric series

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots,$$

which converges for every x between -1 and $+1$, we integrate it term by term between the limits 0 and x, where $|x| < 1$, and thus obtain

$$\int_0^x \frac{dx}{1 + x} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n + 1} + \cdots$$

This series converges for $-1 < x < 1$. The formula is also valid for $x = 1$, for the series converges for $x = 1$.

Hence, we have

$$\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$$

This formula only holds when $x \leqq 1$, so it is not adapted for the computation of logarithms of whole numbers, and in any case it is slowly convergent. A better formula for computation is obtained as follows.

Replace x by $-x$, and we get

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots,$$

which holds for x between -1 and $+1$. Subtracting this from the preceding equation, we have the formula

$$(a) \qquad \log\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

When x varies from 0 to 1, the fraction $\frac{1+x}{1-x}$ steadily increases from 0 to ∞, and hence we may readily calculate from this formula the logarithms of all integers. A still more rapidly convergent series may be obtained by putting $\frac{1+x}{1-x} = \frac{N+1}{N}$, so that $x = \frac{1}{2N+1}$. Then we find

$$(b) \quad \log(N+1) = \log N + 2\left\{\frac{1}{2N+1} + \frac{1}{3}\cdot\frac{1}{(2N+1)^3} + \frac{1}{5}\cdot\frac{1}{(2N+1)^5} + \cdots\right\}.$$

This series converges very rapidly, especially for large values of N.

189. The Binomial Series.—*The series*

$$1 + \frac{m}{1}x + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

is called the **binomial series.**

If m is a positive integer, this series contains only $m + 1$ terms and is a finite series or polynomial, and its sum is $(1 + x)^m$.

If m is not a positive integer, the series is an infinite series. By the ratio test we find that it converges absolutely for $|x| < 1$, and is non-convergent for $|x| > 1$. To determine whether the series converges for $x = 1$ or $x = -1$ requires special investigation (Art. **190**).

Denote the sum of the series by $f(x)$, for $|x| < 1$. Differentiating this power series term by term, we get

$$f'(x) = m\left\{1 + \frac{m-1}{1}x + \frac{(m-1)(m-2)}{2!}x^2 + \cdots \right.$$
$$\left. + \frac{(m-1)(m-2)\cdots(m-n+1)}{(n-1)!}x^{n-1} + \cdots\right\}.$$

Multiply each side by $1 + x$ and collect terms in like powers of x, and then using the algebraic identity

$$\frac{(m-1)(m-2)\cdots(m-n+1)}{(n-1)!} + \frac{(m-1)\cdots(m-n)}{n!} = \frac{m(m-1)\cdots(m-n+1)}{n!},$$

we find

$$(1 + x)f'(x) = f(x),$$

$$\therefore \frac{f'(x)}{f(x)} = \frac{m}{1+x}.$$

Integrating, we get

$$\log f(x) = m \log (1 + x) + \log C,$$

or

$$f(x) = C(1 + x)^m.$$

To determine the constant C, put $x = 0$, and since $f(0) = 1$, we find $C = 1$. Therefore

$$f(x) = (1 + x)^m.$$

Hence, we have the **binomial theorem:**

The series

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

converges absolutely for $|x| < 1$, *and its sum is* $(1 + x)^m$.

190. Behavior of the Binomial Series at the End-points of Its Interval of Convergence.—For $x = 1$, the binomial power series is

$$1 + m + \frac{m(m-1)}{2!} + \frac{m(m-1)(m-2)}{3!} + \cdots$$

D'Alembert's ratio test gives no information unless $m < -1$. Applying Raabe's test, we find

$$R_n \equiv n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) = \frac{1+m}{1 - \dfrac{1+m}{n}}$$

for n sufficiently large, hence

$$LR_n = 1 + m,$$

and hence the series converges absolutely if $m > 0$, and the series of its absolute values diverges if $m < 0$, so that the binomial series does not converge absolutely for $m < 0$. But in this case the terms of the series are alternately positive and negative, and since

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|1 - \frac{1+m}{n}\right|,$$

the $|a_n|$ form a decreasing sequence from a certain term on, if $m > -1$, when $La_n = 0$. Then the series converges conditionally for $m > -1$. and diverges for $m \leqq -1$.

For $x = -1$, the binomial series is

$$1 - m + \frac{m(m-1)}{2!} - \frac{m(m-1)(m-2)}{3!} + \cdots$$

If $m > 0$, the terms have finally all one sign, and

$$R_n \equiv n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) \to 1 + m,$$

so that the series converges absolutely for $m > 0$. If $m < 0$, let $m = -\lambda$. Then the series becomes

$$1 + \lambda + \frac{\lambda(\lambda + 1)}{2!} + \frac{\lambda(\lambda + 1)(\lambda + 2)}{3!} + \cdots$$

For this series we have

$$R_n = \frac{1 - \lambda}{1 + \frac{\lambda - 1}{n}} \to 1 - \lambda,$$

and hence the series diverges in this case.

Summing up, we have the

Theorem.—*The binomial series converges absolutely for* $|x| < 1$, *and diverges for* $|x| > 1$. *When* $x = 1$, *it converges for* $m > -1$, *and diverges for* $m \leqq -1$; *it converges absolutely only for* $m > 0$, *and diverges for* $m < 0$.

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CHAPTER XV

DOUBLE SEQUENCES

191. Double Sequence.—If the number u_{mn} depends upon the number-pair (m, n), where m and n are positive integers, in such a way that we have a means of determining u_{mn} when m and n are given, then if we give to m and n all positive integral values, we say that these numbers form an infinite **double sequence,** which may be exhibited in the form of a rectangular array:

$$\begin{array}{ccccccc}
u_{11} & u_{12} & u_{13} & u_{14} & \cdots & u_{1n} & \cdots \\
u_{21} & u_{22} & u_{23} & u_{24} & \cdots & u_{2n} & \cdots \\
u_{31} & u_{32} & u_{33} & u_{34} & \cdots & u_{3n} & \cdots \\
u_{41} & u_{42} & u_{43} & u_{44} & \cdots & u_{4n} & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \\
u_{m1} & u_{m2} & u_{m3} & u_{m4} & \cdots & u_{mn} & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots &
\end{array}$$

We may denote it briefly by (u_{mn}).

192. Convergence.—*The double sequence (u_{mn}) is said to be* **convergent** *if u_{mn} approaches a definite limit as m and n tend to infinity simultaneously but independently, and this limit U is called the* **simultaneous double limit,** *or simply the* **double limit** *of the sequence (u_{mn}).* In more precise terms, *the double sequence (u_{mn}) is* **convergent** *with the* **double**

limit *U when there exists a number U such that for every* $\epsilon > 0$, *as small as we please, two positive integral indices M and N can be found such that for every* $m > M$, $n > N$, *we have*

$$|U - u_{mn}| < \epsilon.$$

In this case we write

$$U = \underset{m,n}{L} u_{mn}.$$

Some writers use the notation

$$\lim_{m,n \to \infty} u_{mn} \text{ or } \lim_{\substack{m = \infty \\ n = \infty}} u_{mn}.$$

Evidently, instead of the two inequalities $m > M$, $n > N$, we may use $m > P$, $n > P$, or more briefly, $m, n > P$.

It is of the greatest importance to remember that in this definition, m and n must $\to \infty$ **simultaneously** but **independently.** We cannot let $m \to \infty$ first and then let $n \to \infty$, according to this definition.

193. Divergence.—The double sequence (u_{mn}) is said to be divergent if u_{mn} increases beyond all bounds numerically as m and n tend to infinity simultaneously but independently. In other words, *a double sequence* (u_{mn}) *is* **divergent** *when, for any arbitrarily large positive number G, it is possible to assign two positive integral indices M and N such that for every* $m > M$, $n > N$, *we have*

$$|u_{mn}| > G.$$

If for $m > M$, $n > N$, the u_{mn} are either all positive or all negative, and $|u_{mn}| > G$, the sequence is **properly divergent,** and we write

$$\underset{m,n}{L} u_{mn} = +\infty \text{ or } \underset{m,n}{L} u_{mn} = -\infty.$$

When the sequence is divergent, but not properly divergent, it is called **improperly divergent.**

194. Oscillation.—*If the double sequence* (u_{mn}) *is neither convergent nor divergent, it is called* **oscillating.**

195. General Condition for Convergence. Theorem.—*The necessary and sufficient condition for the convergence of the double sequence* (u_{mn}) *is that, for any arbitrarily small* $\epsilon > 0$, *a positive integral index* P *can be found such that for every* $m > P, n > P$, *and for every* p *and* q, *we have*

$$\left|u_{m+p,\ n+q} - u_{m,n}\right| < \epsilon.$$

The proof of this theorem can be carried out in a manner quite similar to that of Art. **26** for simple sequences.

196. Examples of Double Sequences.—(1) If $u_{mn} = \dfrac{1}{m} + \dfrac{1}{n}$, then $\underset{m,n}{L} u_{mn} = 0$, and the sequence is convergent.

(2) If $u_{mn} = m + n$, then $\underset{m,n}{L} u_{mn} = +\infty$, and the sequence is properly divergent.

(3) If $u_{mn} = (-1)^{m+n}$, then u_{mn} has no limit as m and n become infinite simultaneously and independently, and so the sequence is oscillating; it oscillates between -1 and $+1$.

197. Row and Column Sequences. Diagonal Sequence. *The simple sequence*

$$u_{m1},\ u_{m2},\ u_{m3},\ \ .\ \ .\ \ .\ \ ,\ u_{mn},\ \ .\ \ .\ \ .$$

consisting of the elements of the m^{th} *row of the rectangular array of Art.* **191,** *is called a* **row-sequence** *of the double sequence* (u_{mn}), *or more precisely, the* m^{th} **row-sequence.**

The simple sequence

$$u_{1n},\ u_{2n},\ u_{3n},\ \ .\ \ .\ \ .\ \ ,\ u_{mn},\ \ .\ \ .\ \ .$$

made up of the elements of the n^{th} *column of the rectangular array is called a* **column-sequence** *of the double sequence, or the* n^{th} **column-sequence.**

The simple sequence

$$u_{11}, u_{22}, u_{33}, \quad . \quad . \quad . \quad, u_{nn}, \quad . \quad . \quad .$$

consisting of the elements in the principal diagonal of the rectangular array, is called the **diagonal-sequence** *of the double sequence.*

198. Repeated Double Limits.—Suppose that the m^{th} row-sequence

$$u_{m1}, u_{m2}, u_{m3}, \quad . \quad . \quad . \quad, u_{mn}, \quad . \quad . \quad . \quad,$$

regarded as a simple sequence with m fixed, is convergent with limit u_m: $\underset{n}{L} u_{mn} = u_m$. Now suppose that the u_m for $m = 1, 2, 3, \ldots$ form a simple sequence which converges to a limit U, then we may write

$$U = \underset{m}{L} u_m = \underset{m}{L}(\underset{n}{L} u_{mn}) \equiv \underset{m}{L}\ \underset{n}{L} u_{mn}.$$

This limit $\underset{m}{L}\ \underset{n}{L} u_{mn}$ *is called a* **repeated double limit.**

Similarly, starting with column-sequences, we arrive at the **repeated double limit** $\underset{n}{L}\underset{m}{L} u_{mn}$.

We must not assume that these two repeated double limits are necessarily equal, as they are only equal under certain conditions. That is, we cannot interchange the order in a double repeated limit without specific justification.

If the double sequence (u_{mn}) is convergent, we cannot conclude that its row- and column-sequences will also be convergent; we cannot therefore conclude that when the simultaneous double limit $\underset{m,n}{L} u_{mn}$ exists, the simple limits $\underset{m}{L} u_{mn}$, $\underset{n}{L} u_{mn}$ will exist for finite values of n, or m, respectively. This is shown by means of the following example:

If we have

$$u_{mn} = (-1)^{m+n}\left(\frac{1}{m^\alpha} + \frac{1}{n^\beta}\right)$$

for m and n positive integers, and $\alpha > 0$, $\beta > 0$, then we find

$$\underset{m,n}{L} u_{mn} = 0,$$

while for $n = 1, 2, 3, \ . \ . \ . \ ,$

$$\underset{m}{\underline{L}} u_{mn} = -\frac{1}{n^{\beta}}, \ \underset{m}{\overline{L}} u_{mn} = +\frac{1}{n^{\beta}},$$

and for $m = 1, 2, 3, \ . \ . \ . \ ,$

$$\underset{n}{\underline{L}} u_{mn} = -\frac{1}{m^{\alpha}}, \ \underset{n}{\overline{L}} u_{mn} = +\frac{1}{m^{\alpha}},$$

so that $\underset{m}{L} u_{mn}$ and $\underset{n}{L} u_{mn}$ do not exist in this case.

199. Theorem.—*If the double sequence* (u_{mn}) *is either convergent or properly divergent, then we have always*

$$\underset{n}{L}\,\underset{m}{\underline{L}} u_{mn} = \underset{n}{L}\,\underset{m}{\overline{L}} u_{mn} = \underset{m}{L}\,\underset{n}{\underline{L}} u_{mn} = \underset{m}{L}\,\underset{n}{\overline{L}} u_{mn} = \underset{m,n}{L} u_{mn}.$$

We may write this

$$\underset{m,n}{L} = \underset{m}{L}\,\underset{n}{\overline{\underline{L}}} = \underset{n}{L}\,\underset{m}{\overline{\underline{L}}}.$$

For, suppose first that (u_{mn}) is convergent with the simultaneous double limit U, then we have, with the usual notation,

$$|u_{mn} - U| < \epsilon, \ m \geqq M, \ n \geqq N,$$

whence

$$|\underset{m}{\underline{L}} u_{mn} - U| < \epsilon, \ n \geqq N, \ |\underset{m}{\overline{L}} u_{mn} - U| < \epsilon,$$

which gives us

$$\underset{n}{L}\,\underset{m}{\underline{L}} u_{mn} = \underset{n}{L}\,\underset{m}{\overline{L}} u_{mn} = U.$$

Similar considerations apply to $\underset{n}{\underline{L}} u_{mn}$ and $\underset{n}{\overline{L}} u_{mn}$.

If (u_{mn}) is properly divergent, we have either $u_{mn} > G$ or $u_{mn} < -G$ for $m \geqq M$, $n \geqq N$, from which we get

$$\underset{m}{\underline{L}}\, u_{mn} > G,\ \underset{m}{\overline{L}}\, u_{mn} > G, \text{ or } \underset{m}{\underline{L}}\, u_{mn} < -G,\ \underset{m}{\overline{L}}\, u_{mn} < -G$$

for $n \geqq N$, so that again

$$\underset{n}{L}\,\underset{m}{\underline{L}}\, u_{mn} = \underset{n}{L}\,\underset{m}{\overline{L}}\, u_{mn} = +\infty \text{ or } -\infty,$$

and similarly for $\underset{n}{\underline{L}}\, u_{mn}$ and $\underset{n}{\overline{L}}\, u_{mn}$.

200. Theorem.—*If* $\underset{m,n}{L}\, u_{mn}$ *exists, and* $\underset{m}{L}\, u_{mn}$ *exists for* $n \geqq N$, *or* $\underset{n}{L}\, u_{mn}$ *exists for* $m \geqq M$, *either finite or properly infinite, then we have*

$$\underset{n}{L}\,\underset{m}{L}\, u_{mn} = \underset{m,n}{L}\, u_{mn},$$

or

$$\underset{m}{L}\,\underset{n}{L}\, u_{mn} = \underset{m,n}{L}\, u_{mn}.$$

This follows from Art. **199** by taking $\underset{m}{\underline{L}}\, u_{mn} = \underset{m}{\overline{L}}\, u_{mn}$ for $n \geqq N$, or $\underset{n}{\underline{L}}\, u_{mn} = \underset{n}{\overline{L}}\, u_{mn}$ for $m \geqq M$.

From this result it follows that if the limits

$$\underset{m,n}{L}\, u_{mn},\ \underset{m}{L}\, u_{mn}\ (n \geqq N),\ \underset{n}{L}\, u_{mn}\ (m \geqq M)$$

exist, finite or properly infinite, then

$$\underset{m}{L}\,\underset{n}{L}\, u_{mn},\ \underset{n}{L}\,\underset{m}{L}\, u_{mn}$$

exist, and the relation

$$(a) \qquad \underset{m}{L}\,\underset{n}{L}\, u_{mn} = \underset{n}{L}\,\underset{m}{L}\, u_{mn}$$

holds.

Hence the simultaneous double limit $\underset{m,n}{L}\, u_{mn}$ cannot exist, if the two repeated double limits $\underset{m}{L}\,\underset{n}{L}\, u_{mn}$, $\underset{n}{L}\,\underset{m}{L}\, u_{mn}$ both exist but are not equal.

Example.—(1) If $u_{mn} = \frac{m}{m+n}$, we have for every n,

$$\underset{m}{L} u_{mn} = 1,$$

and for every m,

$$\underset{n}{L} u_{mn} = 0,$$

so that

$$\underset{n}{L}\underset{m}{L} u_{mn} = 1, \underset{m}{L}\underset{n}{L} u_{mn} = 0,$$

and equation (a) is not satisfied. Our double sequence (u_{mn}) can therefore be neither convergent nor properly divergent, as can be easily verified immediately.

From the relation (a), which implicitly assumes the existence of both the simple limits $\underset{m}{L} u_{mn}$, $\underset{n}{L} u_{mn}$, we cannot conclude the existence of the double limit $\underset{m,n}{L} u_{mn}$.

Examples.—(2) If $u_{mn} = \frac{1}{1+(m-n)^2}$, we have for every m, $\underset{n}{L} u_{mn} = 0$, and for every n, $\underset{m}{L} u_{mn} = 0$, so that the two repeated double limits are equal, and yet $\underset{m,n}{L} u_{mn}$ cannot exist, for $m - n$ is completely indeterminate as m and n $\rightarrow \infty$ simultaneously and independently. But we have for every m and n,

$$0 \leqq u_{mn} \leqq 1.$$

(3) If

$$u_{mn} = \left(\frac{2}{1+(m-n)^2}\right)^{m+1},$$

we have $\underset{n}{L} u_{mn} = 0$ for every m, and $\underset{m}{L} u_{mn} = 0$ for every n, and yet $u_{mm} = 2^{m+1}$ which $\rightarrow \infty$.

201. Monotonic Double Sequences.—*The double sequence* (u_{mn}) *is called* **monotonic increasing** *if for every positive integral value of* p, q, *we have*

$$u_{m+p,\, n+q} - u_{m,n} \geqq 0,$$

and **monotonic decreasing** *if*

$$u_{m+p,n+q} - u_{m,n} \leqq 0.$$

202. Theorem.—*If* $|u_{mn}| < g$ *(a finite constant), and* $(u_{m,n})$ is *monotonic, then the limits* $\underset{m}{L}u_{mn}$ *(for every* n*) and* $\underset{n}{L}u_{mn}$ *(for every* m*), and* $\underset{m,n}{L}u_{mn}$ *exist with finite values, and we have the relations*

$$\underset{m,n}{L}u_{mn} = \underset{m}{L}\,\underset{n}{L}u_{mn} = \underset{n}{L}\,\underset{m}{L}u_{mn}.$$

Suppose first that the sequence is monotonic increasing, then we have for every r,

$$u_{m+r,n} - u_{m,n} \geqq 0,\ u_{m,n+r} - u_{m,n} \geqq 0,$$

which shows that the row- and column-sequences of the double sequence (u_{mn}) form monotonic increasing simple sequences which are bounded, and the existence of the first two simple limits follows at once from Art. **28.**

Now if the double limit $\underset{m,n}{L}u_{mn}$ did not exist, finite and determinate, for every term u_{mn}, however great m and n might be taken, another term $u_{m_1n_1}$ $(m_1 > m,\ n_1 > n)$ must always exist such that $u_{m_1n_1} - u_{mn} \geqq \alpha$, where α is a given positive number $\neq 0$. Then we can proceed in the same way and determine the positive integers m_2, n_2; m_3, n_3; m_4, n_4; . . . , so that

$$u_{m_2n_2} - u_{m_1n_1} \geqq \alpha,\ u_{m_3n_3} - u_{m_2n_2} \geqq \alpha,\ . \ . \ . \ ,$$
$$u_{m_pn_p} - u_{m_{p-1}n_{p-1}} \geqq \alpha,$$

and adding these inequalities, we obtain

$$u_{m_pn_p} \geqq u_{mn} + p\alpha.$$

Since p can be taken as large as we please, we see that $u_{m_pn_p}$ must $\to \infty$, which contradicts our hypothesis that u_{mn} is bounded: $|u_{mn}| < g$, hence $\underset{m,n}{L} u_{mn}$ must exist with a finite value.

If (u_{mn}) were monotonic decreasing, the double sequence $(-u_{mn})$ would be monotonic increasing, so that the results above would apply in this case.

203. Theorem.—*If the double sequence* (u_{mn}) *is monotonic, then the existence of any one of the three limits*

$$\underset{m,n}{L} u_{mn},\ \underset{m}{L}\underset{n}{L} u_{mn},\ \underset{n}{L}\underset{m}{L} u_{mn}$$

(finite or infinite) implies the existence of the other two and the equality of all three limits.

For, from any one of the equations

$$\underset{m}{L}\underset{n}{L} u_{mn} = U,\ \underset{n}{L}\underset{m}{L} u_{mn} = U,\ \underset{m,n}{L} u_{mn} = U,$$

we obtain, for every m and n,

$$u_{11} \leqq u_{mn} \leqq U, \text{ or } u_{11} \geqq u_{mn} \geqq U,$$

according as (u_{mn}) is increasing or decreasing; in both cases all the absolute values $|u_{mn}|$ must remain less than a finite constant, and our theorem follows at once from Art. **202.**

This proof can be easily modified to apply to the case of infinite limits.

204. Theorem.—*If the double sequence* (u_{mn}) *is monotonic, and its diagonal sequence converges with the limit* U, *then the double sequence is itself convergent with the same limit* U *and we may write*

$$\underset{r}{L} u_{rr} = \underset{m,n}{L} u_{mn} = U.$$

We can evidently restrict ourselves to a monotonic increasing double sequence. If r denotes the greatest of the indices m and n, we have always

$$u_{mn} \leqq u_{rr} \leqq U,$$

and our proposition follows at once from Art. **202.**

205. Continuity of Series of Functions and Double Limits.—If $U(x)$ is the sum of the series of functions $\sum_{1}^{\infty} u_n(x)$, we have

$$U(x) = \underset{n}{L}\, U_n(x),$$

where $U_n(x) = \sum_{i=1}^{n} u_i(x)$. The function $U(x)$ will be continuous at the point $x = x_0$ if

$$\underset{x \to x_0}{L}\, U(x) = U(x_0).$$

But

$$U(x_0) = \underset{n}{L}\left\{ \underset{x \to x_0}{L}\, U_n(x) \right\},$$

and

$$\underset{x \to x_0}{L}\, U(x) = \underset{x \to x_0}{L}\left\{ \underset{n}{L}\, U_n(x) \right\},$$

so that the question as to the continuity of the sum-function $U(x)$ reduces to the question as to the permissibility of the change of order in a double limit: $U(x)$ will be continuous at $x = x_0$, if

$$(a) \qquad \underset{x \to x_0}{L}\left\{ \underset{n}{L}\, U_n(x) \right\} = \underset{n}{L}\left\{ \underset{x \to x_0}{L}\, U_n(x) \right\}.$$

Example.—If $U_n(x) = x^{\frac{2}{2n-1}}$, $x_0 = 0$, then

$$\underset{n}{L}\, U_n(x) = 1 \ (x \neq 0),$$

so that the left-hand side of equation (a) has the value 1. On the other hand,

$$\underset{x \to x_0}{L}\, U_n(x) = U(x_0) = 0,$$

and the right-hand side of (a) has the value 0, so that we may not change the order in the double limit in this case without changing its value.

206. Integration of Series of Functions and Double Limits.—The integration of a series of functions term-wise involves the problem of inversion of order in a double limit. For

$$\int_{x_0}^{x_1} U(x)dx = \int_{x_0}^{x_1} \left\{ \underset{n}{L} U_n(x) \right\} dx,$$

and

$$\sum_{n=1}^{\infty} \int_{x_0}^{x_1} u_n(x)dx = \underset{n}{L} \int_{x_0}^{x_1} U_n(x)dx$$

We have then to ask: When is

$$(a) \qquad \int_{x_0}^{x_1} \left\{ \underset{n}{L} U_n(x) \right\} dx = \underset{n}{L} \left\{ \int_{x_0}^{x_1} U_n(x)dx \right\} ?$$

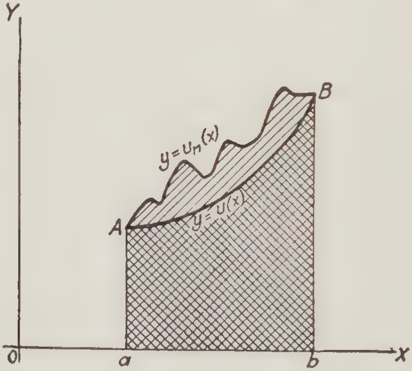

FIG. 7.

And when we consider that the integral is itself a limit, we see that it is again a question of changing the order in a double limit.

Geometrically, the left-hand side of (a) represents the area under the curve $y = U(x)$, while the right-hand side represents the area under the approximation curve $y = U_n(x)$. The question of the integration of the series then reduces to that of when the area under the approximation curves will approach that under the limit-curve $y = U(x)$.

Example.—If

$$U_n(x) = nxe^{-nx^2},$$

then

$$U(x) = \sum_{n=1}^{\infty} \left\{ nxe^{-nx^2} - (n-1)xe^{-(n-1)x^2} \right\} = 0,$$

and if we put $x_0 = 0$, we have

$$\int_0^{x_1} U(x)dx = 0.$$

But

$$\int_0^{x_1} U_n(x)dx = \int_0^{x_1} nxe^{-nx^2}dx = \tfrac{1}{2}(1 - e^{-nx_1^2}),$$

$$\therefore \underset{n}{L} \int_0^{x_1} U_n(x)dx = \tfrac{1}{2}.$$

Thus equation (a) is not satisfied.

207. Differentiation of Series of Functions and Double Limits.—Here again we are concerned with the problem of the inversion of order in a double limit. For

$$U'(x_0) = \underset{\Delta x \to 0}{L} \left\{ \underset{n}{L} \frac{U_n(x_0 + \Delta x) - U_n(x_0)}{\Delta x} \right\},$$

$$\sum_1^{\infty} u'_n(x) = \underset{n}{L} \left\{ \underset{\Delta x \to 0}{L} \frac{U_n(x_0 + \Delta x) - U_n(x_0)}{\Delta x} \right\},$$

and the question is: When is

$$(a) \quad \underset{\Delta x \to 0}{L} \left\{ \underset{n}{L} \frac{U_n(x_0 + \Delta x) - U_n(x_0)}{\Delta x} \right\} = \underset{n}{L} \left\{ \underset{\Delta x \to 0}{L} \frac{U_n(x_0 + \Delta x) - U_n(x_0)}{\Delta x} \right\} ?$$

Geometrically, the left-hand side of (*a*) represents the slope of the tangent to the limit-curve $y = U(x)$ at the point $x = x_0$, while on the right-hand side we have to do with the slope of the tangent to the approximation-curves $y = U_n(x)$ at x_0. The question to decide then is whether the tangents to the approximation curves at x_0 will approach the tangent to the curve $y = U(x)$ at x_0 as we proceed through the approximation curves. By a few simple examples, we can see that this will not always be the case.

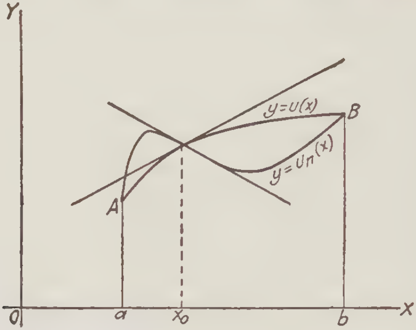

FIG. 8.

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CHAPTER XVI

DOUBLE AND MULTIPLE SERIES

208. Double Series.—*The doubly infinite rectangular array*

$$
(a)\qquad
\begin{array}{ccccccccccc}
 & u_{11} & + & u_{12} & + & u_{13} & + & u_{14} & + & \cdots \\
+ & u_{21} & + & u_{22} & + & u_{23} & + & u_{24} & + & \cdots \\
+ & u_{31} & + & u_{32} & + & u_{33} & + & u_{34} & + & \cdots \\
+ & u_{41} & + & u_{42} & + & u_{43} & + & u_{44} & + & \cdots \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

is called a **double series.** This rectangular array extends to infinity to the right and below, and in each term the first subscript indicates the number of the row, and the second subscript indicates the number of the column in which that term stands. We shall denote the double series (*a*) briefly by $\sum\limits_{m,n} u_{mn}$, or by $\sum u_{mn}$ if no confusion is likely to arise.

Denote by U_{mn} the sum of all the terms which are contained in the rectangle which is formed by the first m rows and n columns:

$$
\begin{array}{rcccccccccc}
U_{mn} = & u_{11} & + & u_{12} & + & u_{13} & + & \cdots & + & u_{1n} \\
+ & u_{21} & + & u_{22} & + & u_{23} & + & \cdots & + & u_{2n} \\
+ & u_{31} & + & u_{32} & + & u_{33} & + & \cdots & + & u_{3n} \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
+ & u_{m1} & + & u_{m2} & + & u_{m3} & + & \cdots & + & u_{mn}.
\end{array}
$$

Then we have associated with the double series the double sequence (U_{mn}). The numbers U_{mn} may be called the **partial sums.**

209. Convergence.—*The double series* $\sum_{m,n} u_{mn}$ *is said to be* **convergent** *when the double sequence* (U_{mn}) *is convergent, and the simultaneous double limit* U *of the sequence* (U_{mn}) *is called the* **sum** *of the series.*

Then we may write

$$|U - U_{mn}| < \epsilon,$$

however small ϵ may be, for every $m > M$, $n > N$.

210. Divergence.—*The double series* $\sum_{m,n} u_{mn}$ *is called* **divergent** *when the double sequence* (U_{mn}) *is divergent, and is* **properly** *or* **improperly divergent** *according as the sequence* (U_{mn}) *is.*

Then for every divergent double series, we have

$$|U_{mn}| > G,$$

however large $G > 0$ may be, for $m > M$, $n > N$; and if the series is properly divergent, the above inequality becomes

$$U_{mn} > G \text{ or } < -G.$$

211. Oscillation.—*If the double series* $\sum_{m,n} u_{mn}$ *is neither convergent nor divergent, it is called* **oscillating,** *and the sequence* (U_{mn}) *is also oscillating in this case.*

212. General Condition for Convergence. Theorem.—*The necessary and sufficient condition for convergence of the double series* $\sum_{m,n} u_{mn}$ *is that, for any* $\epsilon > 0$, *however small, a positive integral index* P *can be found such that for every* $m > P$, $n > P$, *and for every* $p, q = 1, 2, 3, \ldots$, *we have*

$$|U_{m+p,n+q} - U_{m,n}| < \epsilon.$$

This follows at once from Art. **195**.

213. Formulas for Partial Sums.—From the definition of U_{mn}, we find

$$(a) \qquad U_{1n} = u_{11} + u_{12} + u_{13} + \cdot \cdot \cdot + u_{1n},$$

$$(b) \qquad U_{m1} = u_{11} + u_{21} + u_{31} + \cdot \cdot \cdot + u_{m1},$$

$$(c) \qquad U_{m,n} - U_{m,n-1} = u_{1n} + u_{2n} + u_{3n} + \cdot \cdot \cdot + u_{mn} \quad (n \geqq 2),$$

$$(d) \qquad U_{m,n} - U_{m-1,n} = u_{m1} + u_{m2} + u_{m3} + \cdot \cdot \cdot + u_{mn} \quad (m \geqq 2),$$

hence for $m \geqq 2$, $n \geqq 2$,

$$(e) \qquad u_{11} = U_{11},$$

$$(f) \qquad u_{m1} = U_{m1} - U_{m-1,1},$$

$$(g) \qquad u_{1n} = U_{1n} - U_{1,n-1},$$

$$(h) \qquad u_{mn} = U_{mn} - U_{m,n-1} - (U_{m-1,n} - U_{m-1,n-1}),$$

$$(i) \qquad u_{mn} = U_{mn} + U_{m-1,n-1} - (U_{m-1,n} + U_{m,n-1}),$$

$$(j) \qquad u_{mn} = U_{mn} - U_{m,n-1} - U_{m-1,n} + U_{m-1,n-1}.$$

214. Necessary Condition for Convergence.—Since

$$u_{mn} = U_{mn} - U_{m,n-1} - U_{m-1,n} + U_{m-1,n-1},$$

we see that by our general condition for convergence (Art. **212**), when the double series $\sum u_{mn}$ is convergent, we can find an index P such that $|u_{mn}| < \epsilon$ for every m, $n > P$.

Theorem.—*The necessary condition for convergence of a double series* $\sum u_{mn}$ *is*

$$\underset{m,n}{L}\, u_{mn} = 0.$$

This condition is, however, not sufficient.

As a special case, for $m = n$, for the terms of the principal diagonal,

$$\underset{r}{L}\, u_{rr} = 0.$$

215. Repeated Summation.—The simple infinite series

$$\sum_{j=1}^{\infty} u_{mj} = u_{m1} + u_{m2} + \cdot \cdot \cdot + u_{mj} + \cdot \cdot \cdot \qquad (m = 1, 2, 3, \cdot \cdot \cdot),$$

$$\sum_{i=1}^{\infty} u_{in} = u_{1n} + u_{2n} + \ . \ . \ . \ + u_{in} + \ . \ . \ .$$
$$(n = 1, 2, 3, \ . \ . \ . \ )$$

are called, respectively, the **row-** and **column-series.** If they are convergent, we shall denote their sums by r_m and c_n respectively.

If every row-series r_m is convergent $(m = 1, 2, 3, \ . \ . \ .)$, and we form the series $\sum_{m=1}^{\infty} r_m$, and if this series is convergent, we get a value, the sum of $\sum r_m$, which is called the **sum by rows** of the double series, and the process is called **summation by rows.** It will be represented by

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} u_{mn}\right), \text{ or } \sum_{m}\sum_{n} u_{mn}.$$

Similarly, if each column-series c_n is convergent $(n = 1, 2, 3, \ . \ . \ .)$, and the series $\sum_{n=1}^{\infty} c_n$ is formed and is found to be convergent, we obtain a value called the **sum by columns,** and the process is called **summation by columns;** it will be represented by

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} u_{mn}\right), \text{ or } \sum_{n}\sum_{m} u_{mn}.$$

This repeated summation corresponds to taking repeated limits; thus

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} u_{mn}\right) = \underset{m}{L}\left(\underset{n}{L}\ U_{mn}\right),$$
$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} u_{mn}\right) = \underset{n}{L}\left(\underset{m}{L}\ U_{mn}\right).$$

216. Theorem.—*A double series may converge without the row- or column-series being convergent.*

For a finite number of terms, we have

$$\sum_{m=1}^{M}\left(\sum_{n=1}^{N} u_{mn}\right) = \sum_{n=1}^{N}\left(\sum_{m=1}^{M} u_{mn}\right).$$

But this is not necessarily true for an infinite number of terms, for the row- or column-series may not converge but may oscillate. This is shown in Art. **198**.

217. Theorem.—*A double series can converge without any of its terms, in which one subscript* $\rightarrow \infty$, *while the other remains finite, approaching zero.*

This follows at once from Arts. **213, 216.**

Example.—If

$$U_{mn} = \frac{(-1)^{m+n}}{a+1}\left(\frac{1}{a^m} + \frac{1}{a^n}\right) \qquad (a > 1),$$

the corresponding double series is convergent with the sum 0, since $\underset{m,n}{L} U_{mn} = 0$. We find from Art. **213,**

$$u_{1n} = (-1)^n\left(\frac{1}{a^n} + \frac{2}{a+1}\right) \qquad (n \geqq 2),$$

$$u_{m1} = (-1)^m\left(\frac{1}{a^m} + \frac{2}{a+1}\right) \qquad (m \geqq 2),$$

and for all other elements,

$$u_{11} = \frac{2}{a+1}, \; u_{mn} = (-1)^{m+n}\left(\frac{2}{a^m} + \frac{2}{a^n}\right).$$

We have then

$$\underset{m}{L}\left|u_{m1}\right| = \underset{n}{L}\left|u_{1n}\right| = \frac{2}{a+1},$$

and in general for any finite m or n,

$$\underset{m}{L}\left|u_{mn}\right| = \frac{2}{a^n}, \qquad \underset{n}{L}\left|u_{mn}\right| = \frac{2}{a^m}.$$

218. Theorem.—*If the double series* $\sum u_{mn}$ *is convergent with sum* U, *and if the rows and columns converge, then*

$$(a) \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} u_{mn} = U.$$

For since the double series is convergent, we have by definition,

$$|U_{mn} - U| < \epsilon,\ m,n > P,$$

and since $\underset{n}{L} U_{mn}$ exists, by hypothesis, we have therefore

$$|\underset{n}{L} U_{mn} - U| \leqq \epsilon,\ m > P.$$

Hence,

$$\underset{m}{L}\left(\underset{n}{L}\, U_{mn}\right) = U.$$

Similarly, we may prove the other part of equation (a).

The theorem also follows immediately from Art. **200.**

219. Theorem.—*When the double series* $\sum u_{mn}$ *is not convergent, and the two repeated summations exist with finite values, it is not necessarily true that these values are equal, i.e., that*

$$\sum_{m}\sum_{n} u_{mn} = \sum_{n}\sum_{m} u_{mn}.$$

For, if we take for example,

$$U_{mn} = \frac{m}{m+n},$$

we have

$$\underset{m}{L}\ \underset{n}{L}\ U_{mn} = 0,\ \underset{n}{L}\ \underset{m}{L}\ U_{mn} = 1,$$

while U_{mn} is indeterminate as m and n become infinite simultaneously.

From the theorem of Art. **218**, it follows that if the rows and columns converge, the double series cannot converge unless

$$(a) \qquad \sum_m \sum_n u_{mn} = \sum_n \sum_m u_{mn}.$$

However, the truth of (a) is not sufficient to enable us to say that the double series is convergent (Art. **200**). This may also be shown by the example,

$$U_{mn} = \frac{mn}{(m+n)^2},$$

for which

$$\underset{m}{L}\,\underset{n}{L}\,U_{mn} = 0 = \underset{n}{L}\,\underset{m}{L}\,U_{mn},$$

but the double series is not convergent, for if $m = 2n$, then $U_{mn} = 2/9$, and if $m = n$, then $U_{mn} = 1/4$.

220. Theorem.—*If the double series* $\sum u_{mn}$ *converges with sum* U, *and if all the row-series or all the column-series converge, then the corresponding repeated summation, by rows or by columns, exists with the value* U; *that is,*

$$\sum_m \sum_n u_{mn} = U, \text{ or } \sum_n \sum_m u_{mn} = U.$$

This follows at once from Art. **200**.

221. Theorem.—*If the double series* $\sum u_{mn}$ *is properly divergent, but the row-series or the column-series convergent, then the corresponding repeated summation, by rows or by columns, is properly divergent.*

This is an immediate consequence of Art. **200**.

222. Theorem.—*When all the row-series or all the column-series are convergent, and when the repeated summations converge, with the same value* U, *the double series may even diverge, and indeed improperly, or oscillate; this must be the case when the repeated summations give different values.*

This follows at once from Art. **221** and Art. **200**.

Examples.—(1) If

$$U_{mn} = \frac{1}{1-(m-n)^2},$$

we have

$$\underset{m}{L} U_{mn} = \underset{n}{L} U_{mn} = 0,$$
$$\underset{m\ n}{LL} U_{mn} = \underset{n\ m}{LL} U_{mn} = 0,$$

while $\underset{m,\ n}{L} U_{mn}$ does not exist; the double series oscillates.

(2) If

$$U_{mn} = \frac{m+1}{m+n+1},$$

we have

$$\underset{m\ n}{LL} U_{mn} = 0, \ \underset{n\ m}{LL} U_{mn} = 1.$$

223. Theorem.—*If the terms of the double series* $\sum u_{mn}$ *are all positive, and if* U_{mn} *remains always less than a finite number* g, *the double series converges to a definite sum* U; *each row-series and column-series converges and the repeated summations converge to the same sum* U.

This is a result of Art. **202,** since (U_{mn}) is here a monotonic increasing sequence.

224. Theorem.—*If the terms of the double series* $\sum a_{mn}$ *are all positive, the convergence of any one of the three series*

$$\sum_{m,n} a_{mn}, \ \sum_{m} \sum_{n} a_{mn}, \ \sum_{n} \sum_{m} a_{mn}$$

implies the convergence of the other two and the equality of their sums. If one of the three series is properly divergent, the other two must also be.

This is obtained by applying Art. **203.**

225. Diagonal-summation.—If we put

$$w_n = u_{1n} + u_{2,\,n-1} + u_{3,\,n-2} + \ . \ . \ . \ + u_{p,\,n-p} + \ . \ . \ . \ + u_{n-1,\,2} + u_{n1},$$

the series

$$\sum_{n=1}^{\infty} w_n = w_1 + w_2 + \ . \ . \ . \ + w_n + \ . \ . \ .$$

is called the **diagonal-series** of the double series $\sum\limits_{m,n} u_{mn}$. In case the series $\sum w_n$ converges, we denote its sum by W.

226. Theorem.—*If* $a_{mn} \geqq 0$, *the double series* $\sum a_{mn}$ *and the diagonal series* $\sum w_n$ *must either both converge with the same sum, or both diverge.*

From an inspection of the rectangular array formed by the double series, we obtain for $n \geqq 2$,

$$(a) \qquad \sum_{i=1}^{n} w_i \leqq A_{nn} \leqq \sum_{i=1}^{2n} w_i.$$

Now if the double series is convergent with sum A, we have $A_{nn} \leqq A$, and from (a) we see that the positive series $\sum w_n$ is therefore convergent. If we write the relation (a) in the form

$$(b) \qquad 0 \leqq A_{nn} - \sum_{i=1}^{n} w_i \leqq \sum_{i=n+1}^{2n} w_i,$$

we get

$$\sum_{i=1}^{\infty} w_i = A = \sum_{m,n} a_{mn}.$$

On the other hand, if $\sum w_n$ is convergent with sum W, we have by (b),

$$\underset{n}{L} A_{nn} = W,$$

from which, by Art. **204,** we obtain

$$\sum_{m,n} a_{mn} = \underset{m,n}{L} A_{mn} = W.$$

If one of the series $\sum_{m,n} a_{mn}$, $\sum w_n$ diverges, it is easily seen from (a) that the other must also be divergent.

227. Theorem.—*If* $a_{mn} \geqq 0$, *each of the four equations*

$$\sum_{m,n} a_{mn} = A, \ \sum_{n} w_n = A,$$
$$\sum_{m} \sum_{n} a_{mn} = A, \ \sum_{n} \sum_{m} a_{mn} = A,$$

carries with it the other three, whether A is finite or infinite.

This is obtained by combining Arts. **226, 224**.

228. Theorem.—*If the double series $\sum_{m,n} |u_{mn}|$ is convergent, then $\sum_{m,n} u_{mn}$ is also convergent.*

We have

$$\left|\sum_{m,n} u_{mn}\right| \leqq \sum_{m,n} |u_{mn}|,$$

if the summation is extended over the same values of m and n in both cases. Then if U'_{mn} denotes the sum of the absolute values $|u_{mn}|$ of the terms u_{mn} which appear in U_{mn}, we have

$$|U_{m+p,\ n+q} - U_{m,\ n}| \leqq U'_{m+p,\ n+q} - U'_{m,\ n}.$$

But by the general condition for convergence, since $\sum_{m,n} |u_{mn}$ is convergent,

$$U'_{m+p,\ n+q} - U'_{m,\ n} < \epsilon$$

for $m > M$, $n > N$, and every p, q, hence

$$|U_{m+p,\ n+q} - U_{m,n}| < \epsilon$$

for $m > M$, $n > N$, $p, q = 1, 2, 3, \ldots$, which shows that $\sum_{m,n} u_{mn}$ is convergent.

229. Absolute Convergence.—*The double series* $\sum\limits_{m,n} u_{mn}$ *is called* **absolutely convergent** *when the double series* $\sum\limits_{m,n} |u_{mn}|$ *formed from the absolute values of its terms converges.*

Absolutely convergent double series are very similar in their properties to the convergent double series with positive terms.

230. Theorem.—*If the double series* $\sum\limits_{m,n} u_{mn}$ *is absolutely convergent with sum* U, *then each row- and column-series converges absolutely, and the repeated summations by rows and by columns converge absolutely, with*

$$(a) \qquad \sum_{m} \sum_{n} u_{mn} = U, \ \sum_{n} \sum_{m} u_{mn} = U.$$

For since the double series $\sum\limits_{m,n} |u_{mn}|$ is convergent, all the series mentioned in the theorem must be absolutely convergent by Art. **223**. The relation (a) follows at once from Art. **203**.

231. Theorem.—*Each of the three equations*

$$\sum_{m,n} u_{mn} = U, \ \sum_{m} \sum_{n} u_{mn} = U, \ \sum_{n} \sum_{m} u_{mn} = U$$

carries with it the other two, if the corresponding series is absolutely convergent.

This result is obtained by application of Arts. **224, 230.**

232. Theorem.—*If the double series* $\sum\limits_{m,n} u_{mn}$ *has its repeated summation by rows absolutely convergent, with the sum* U, *then all the column-series converge and the repeated summation by columns is convergent with sum* U.

This theorem of Cauchy's is a special case of Art. **231.**

233. Comparison Test for Convergence. Theorem.—*If* $a_{mn} \geqq 0$, $c_{mn} \geqq 0$, *and if* $a_{mn} \leqq c_{mn}$ *for every* m *and* n, *then if* $\sum\limits_{m,n} c_{mn}$ *is convergent,* $\sum\limits_{m,n} a_{mn}$ *is also convergent.*

This can be proved in the same way as the corresponding test for simple series.

234. Integral Test for Double Series.—The integral test for simple series may be generalized so as to apply to double series, giving the following

Theorem.—*Let* $f(x, y)$ *be a function of* x, y, *which is positive for all points* (x, y) *outside a certain closed curve* Γ, *and which steadily diminishes in value as the point* (x, y) *recedes from the origin. Consider the value of the double integral* $\int\int f(x,y)dxdy$, *extended over the ring-shaped region between the curve* Γ *and a variable curve* C *outside* Γ, *which recedes indefinitely in all directions, and then compare with it the double series* $\sum\limits_{m,n} f(m, n)$, *where* m *and* n *may have any positive or negative integral values for which the point* (m, n) *lies outside the fixed curve* Γ. *Then the double series converges if the double integral approaches a limit, and conversely.*

As an example of the application of this test, consider the double series

$$\sum_{m,n} \frac{1}{(m^2 + n^2)^\mu}$$

where m and n may take all integral values from $-\infty$ to $+\infty$ except the values $m = n = 0$. Comparing with the double integral

$$\int\int \frac{dxdy}{(x^2 + y^2)^\mu}$$

extended over the region of the plane outside any circle whose center is at the origin, which has a definite value when $\mu > 1$ and becomes infinite when $\mu \leqq 1$, we see that the given double series is convergent when $\mu > 1$ and divergent when $\mu \leqq 1$.

235. Uniform Convergence of a Double Series of Functions.—Let $\sum\limits_{m,n} u_{mn}$ be a double series whose terms are functions of one or more variables, which converges absolutely for all sets of values of these variables in a region S. Let R_c be the sum of the terms of the double series outside any closed curve C drawn in the plane of the rectangular array formed by the series. *We say the given double series* **converges uniformly** *in the region S, if, corresponding to any $\epsilon > 0$, however small, a closed curve K, not dependent on the values of the variables, can be drawn so that $|R_c| < \epsilon$ for any curve C whatever lying outside of K and for any set of values of the variables in the region S.*

236. Double Power Series.—Consider the power series in two variables $\sum\limits_{m,n} a_{mn}x^my^n$, where the indices m and n may take all integral values from 0 to $+\infty$, and where the coefficients a_{mn} are any real numbers. Then we have the fundamental

Theorem.—*If for a set of values $x = x_0$, $y = y_0$, the terms of this double power series do not exceed a positive constant in absolute value, the series converges absolutely for all values of x and y which satisfy the inequalities $|x| < |x_0|$, $|y| < |y_0|$.*

For, if

$$|a_{mn}x_0{}^my_0{}^n| < M,$$

then

$$|a_{mn}| < \frac{M}{|x_0|^m\ |y_0|^n}$$

for all values of m and n. Then the absolute value of the general term of the double series $\sum a_{mn}x^my^n$ is less than the corresponding term of the double series $\sum M\left|\frac{x}{x_0}\right|^m\left|\frac{y}{y_0}\right|^n$.

But the latter series converges when $|x| < |x_0|$, $|y| < |y_0|$, and its sum is

$$\frac{M}{\left(1 - \left|\frac{x}{x_0}\right|\right)\left(1 - \left|\frac{y}{y_0}\right|\right)}$$

as is seen by taking the sums of the terms by columns and then adding these sums.

Now let r and ρ be two positive numbers for which the double series $\sum |a_{mn}| r^m \rho^n$ converges, and let R be the rectangle formed by the four straight lines $x = r$, $x = -r$, $y = \rho$, $y = -\rho$. For every point inside this rectangle or upon one of its sides no term of the double series $\sum a_{mn} x^m y^n$ exceeds the corresponding term of the series $\sum |a_{mn}| r^m \rho^n$ in absolute value. Hence the double power series converges absolutely inside the rectangle R, which is the form of its region of convergence.

237. Generalization of Double Series.—Just as simple series may be generalized by considering the summation to extend from $-\infty$ to $+\infty$, giving rise to two-way series, so we may similarly generalize double series. Let u_{mn} be an element in which both indices m, n may vary from $-\infty$ to $+\infty$, then the set of all such elements may be arranged in a rectangular array which extends indefinitely in all directions in its plane. Such a series may evidently be divided into four double series of the ordinary type.

238. Multiple Series.—*A series of the type*

$$\sum_{m_1, m_2, \ldots, m_p} u_{m_1, m_2, \ldots, m_p},$$

where the indices $m_1, m_2, \ldots m_p$ *may take on any values from* 0 *to* ∞, *or from* $-\infty$ *to* $+\infty$, *but may be restricted by certain inequalities, is called a* **multiple series of order p,** *or a* **p-tuple series.** *Such a series is* **convergent** *with* **sum U,** *when the partial sums*

$$S_{n_1, n_2, \ldots, n_p} \equiv \sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} \cdots \sum_{m_p=0}^{n_p} u_{m_1, m_2, \ldots, m_p}$$

all have a definite finite limit U however the $n_1, n_2, \ldots n_p$, *become infinite, provided they* → ∞ *independently of each other but increase simultaneously.*

The theorems proved for double series admit of immediate generalization to multiple series of any order.

The Cauchy integral test may be extended to multiple series as follows: *Let* $f(x_1, x_2, \ldots, x_p)$ *be a function of* p *variables, always positive, which decreases to* 0 *with increasing values of the x's outside a closed region R of p-dimensional space. Then the p-tuple infinite series*

$$\sum_{m_1, \ldots, m_p} f(m_1, \ldots, m_p)$$

converges or diverges according as the p-tuple integral

$$\int\int \cdots \int f(x_1, \ldots, x_p) dx_1 \ldots dx_p$$

extended over the whole space outside the region R is convergent or not.

By application of this test, we may show that *the multiple series*

$$\sum_{m_1, \ldots, m_p}{}' \frac{1}{(m_1^2 + m_2^2 + \ldots + m_p^2)^\mu},$$

where the set of values $m_1 = m_2 = \ldots = m_p = 0$ *is excluded, converges if* $2\mu > p$.

A power series

$$\sum_{m_1, \ldots, m_p} a_{m_1, \ldots m_p} x_1^{m_1} x_2^{m_2} \ldots x_p^{m_p}$$

in p variables converges uniformly within any region lying within its region of convergence.

239. Substitution of One Power Series in Another.—Let $z = \sum^{\infty} a_n y^n$ be a real power series with the interval of convergence $(-R, R)$, and $y = \sum_0^{\infty} b_n x^n$ another real power series with the interval of convergence $(-r, r)$. Then we may regard z as a function of x, and the problem at once arises to express z as a power series in x.

Using Cauchy's multiplication theorem (Art. **146**), we may obtain $y^2, y^3, \dots$ as power series in x:

$$
\begin{aligned}
y^2 &= b_0^2 + 2b_0b_1x + (b_1^2 + 2b_0b_2)x^2 + \dots, \\
y^3 &= b_0^3 + 3b_0^2b_1x + 3(b_0b_1^2 + b_0^2b_2)x^2 + \dots, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

which converge absolutely in the interval $(-r, r)$. Substituting these values of $y, y^2, y^3, \dots$ in $z = \sum_0^{\infty} a_n y^n$, we get the double series

$$
(a)\quad
\begin{aligned}
&a_0 + \\
&+ a_1b_0 + a_1b_1x + a_1b_2x^2 + \dots \\
&+ a_2b_0^2 + 2a_2b_0b_1x + a_2(b_1^2 + 2b_0b_2)x^2 + \dots \\
&+ a_3b_0^3 + 3a_3b_0^2b_1x + 3a_3(b_0b_1^2 + b_0^2b_2)x^2 + \dots \\
&+ \dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

If we sum this double series by rows, we get a series whose sum is z, since each row is a term of z. If we sum by columns, we obtain an arrangement according to powers of x, and these two sums will be equal if the double series is absolutely convergent.

Put $|a_n| = \alpha_n$, $|b_n| = \beta_n$, $|x| = \xi$, then the series of absolute values of the terms of the double series (a) will be $\leqq$ the positive double series

$$\begin{array}{ll}
 & \alpha_0 + \\
 & +\alpha_1\beta_0 + \alpha_1\beta_1\xi + \alpha_1\beta_2\xi^2 + \ . \ . \ . \\
(b) & +\alpha_2\beta_0^2 + 2\alpha_2\beta_0\beta_1\xi + \alpha_2(\beta_1^2 + 2\beta_0\beta_2)\xi^2 + \ . \ . \ . \\
 & +\alpha_3\beta_0^3 + 3\alpha_3\beta_0^2\beta_1\xi + 3\alpha_3(\beta_0\beta_1^2 + \beta_0^2\beta_2)\xi^2 + \ . \ . \ . \\
 & + \ .
\end{array}$$

This series summed by rows gives

$$\sum_{n=0}^{\infty} \alpha_n\left(\beta_0 + \sum_{m=1}^{\infty} \beta_m\xi^m\right)^n,$$

which converges when

$$\beta_0 + \sum_1^{\infty} \beta_m\xi^m < R.$$

If ρ is any positive number $< r$, the series $\sum \beta_m\rho^m$ is convergent, and hence the terms $\beta_m\rho^m$ have a finite upper bound M. Then

$$\sum_1^{\infty} \beta_m\xi^m = \sum_1^{\infty} \beta_m\rho^m\left(\frac{\xi}{\rho}\right)^m < M \cdot \sum_1^{\infty}\left(\frac{\xi}{\rho}\right)^m = \frac{M\xi}{\rho - \xi},$$

so that our condition for convergence of (*b*) is

$$\beta_0 + \frac{M\xi}{\rho - \xi} < R.$$

Hence, if $\beta_0 < R$, and $\xi < \dfrac{(R - \beta_0)\rho}{M + R - \beta_0}$, the positive series (*b*) will converge, and consequently the series of absolute values of series (*a*) converges and series (*a*) must converge absolutely.

We have then the

Theorem.—*The substitution of* $y = \sum_0^{\infty} b_nx^n$ *in* $z = \sum_0^{\infty} a_ny^n$

is permissible and z *can be represented as a power series in* x, *if the conditions:*

(1) $|b_0| < R$,

(2) $|x| < \dfrac{(R - |b_0|)\rho}{M + R - |b_0|}$,

where $\rho < r$, $|b_n|\rho^n \leqq M$, *are satisfied.*

In particular, if $b_0 = 0$, the conditions may be replaced by

$$|x| < \frac{\rho R}{M + R}.$$

If $z = \sum a_n y^n$ is convergent for all values of y, the condition $|x| < r$ is sufficient to justify the substitution.

If $b_0 = 0$, the coefficient of x^n in the new series is not an infinite series but terminates, and the first few coefficients are

$$\begin{aligned} &c_0 = a_0,\ c_1 = a_1b_1,\ c_2 = a_1b_2 + a_2b_1^2, \\ &c_3 = a_1b_3 + 2a_2b_1b_2 + a_3b_1^3, \ \ldots\ . \end{aligned}$$

Example.—If

$$\begin{aligned} z &= 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots = e^y, \\ y &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin x, \end{aligned}$$

then $z = e^{\sin x}$. Substituting y in z, we get the double series

$$\begin{array}{lllllll} z = 1 + x & + 0\cdot x^2 & - \frac{1}{6}x^3 & + 0\cdot x^4 & + \frac{1}{120}x^5 & + 0.x^6 & + \cdots \\ & + \frac{1}{2}x^2 & + 0 & - \frac{1}{6}x^4 & + 0 & + \frac{1}{45}x^6 & + \cdots \\ & & + \frac{1}{6}x^3 & + 0 & - \frac{1}{12}x^5 & + 0 & + \cdots \\ & & & + \frac{1}{24}x^4 & + 0 & - \frac{1}{36}x^6 & + \cdots \\ & & & & + \frac{1}{120}x^5 & + 0 & + \cdots \\ & & & & & + \frac{1}{720}x^6 & + \cdots \\ & & & & & & + \cdots \end{array}$$

Summing by columns, we get

$$z = e^{\sin x} = 1 + x + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{15}x^5 - \tfrac{1}{240}x^6 + \ . \ . \ .$$

240. Taylor's Development for Power Series.—Consider the power series $z = P(y) = \sum\limits_0^\infty a_n y^n$, with the interval of convergence $(-R, R)$. Let x be a point in the interval $(-R, R)$, and then choose δ so small that $y = x + h$ also lies in the interval $(-R, R)$, for every value of h such that $|h| < \delta$. Then we have

$$\begin{aligned} z = a_0 &+ a_1(x + h) \\ &+ a_2(x^2 + 2xh + h^2) \\ &+ a_3(x^3 + 3x^2h + 3xh^2 + h^3) \\ &+ \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \end{aligned}$$

Regarding this as a double series, by Art. **239** it may be summed by columns, giving

$$\begin{aligned} P(x + h) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \ . \ . \ . \\ &\qquad + h(a_1 + 2a_2x + 3a_3x^2 + \ . \ . \ .) \\ &\quad + \frac{h^2}{2!}(2a_2 + 2 \cdot 3a_3x + 3 \cdot 4a_4x^2 + \ . \ . \ .) \\ &\quad + \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\ &= P(x) + h \cdot P'(x) + \frac{h^2}{2!}P''(x) + \frac{h^3}{3!}P'''(x) + \ . \ . \ . \end{aligned}$$

This is Taylor's development of a function defined by a power series.

241. Division of Power Series. Theorem.—*Let* $P = \sum\limits_0^\infty a_n x^n$, $a_0 \neq 0$, *be absolutely convergent in the interval* $(-R, R)$, *and let* $Q = a_1x + a_2x^2 + a_3x^3 + \ . \ . \ .$ *be numerically* $< |a_0|$ *in the interval* $(-r, r)$, $r < R$. *Then* $\dfrac{1}{P}$ *can* *be developed in a power series* $\frac{1}{P} = \sum c_n x^n$ *which converges in* $(-r, r)$; *the first coefficient is* $c_0 = \frac{1}{a_0}$.

For we may write

$$\frac{1}{P} = \frac{1}{a_0 + Q} = \frac{1}{a_0} \cdot \frac{1}{1 + \frac{Q}{a_0}}$$

$$= \frac{1}{a_0}\left\{1 - \frac{Q}{a_0} + \frac{Q^2}{a_0{}^2} - \frac{Q^3}{a_0{}^3} + \ . \ . \ .\right\}$$

for every x in $(-r, r)$, and applying Art. **239** our theorem follows.

To find the coefficients of $\sum c_n x^n$, we may multiply out in

$$1 = \sum a_n x^n . \sum c_n x^n,$$

and equate coefficients of like powers of x on each side of the equation.

If we now wished to divide one power series, $P_1(x)$, by another, $P(x)$, we write

$$\frac{P_1(x)}{P(x)} = P_1(x) \cdot \frac{1}{P(x)},$$

and apply the preceding result.

242. Reversion of a Power Series.—A general power series $\sum_0^\infty b_n t^n$ may be written in the form

$$x - a_2 x^2 - a_3 x^3 - \ . \ .$$

which converges for $x = 1$, without loss of generality. For if we suppose that the series

$$v = b_0 + b_1 t + b_2 t^2 + \ . \ . \ . \ , b_0 \neq 0,$$

converges for $t = t_0$, and put

$$t = xt_0,\ u = \frac{v - b_0}{b_1 t_0},$$

the series v becomes

$$(a) \qquad u = x - a_2x^2 - a_3x^3 - \ . \ . \ . \ ,$$

which converges for $x = 1$.

The problem of reversion of series is to develop this function x in a power series in u, valid in some interval about $u = 0$.

Let us put

$$(b) \qquad x = u + c_2u^2 + c_3u^3 + \ . \ . \ . \ ,$$

and try to determine the coefficients c_n so that (b) will satisfy (a) formally. Raising (b) to successive powers, we get

$$\begin{aligned}
x^2 &= u^2 + 2c_2u^3 + (c_2^2 + 2c_3)u^4 + (2c_4 + 2c_2c_3)u^5 + \ . \ . \ . , \\
x^3 &= v^3 + 3c_2u^4 + (3c_2^2 + 3c_3)u^5 + \ . \ . \ . \ , \\
x^4 &= u^4 + 4c_2u^5 + \ . \ . \ . \ , \\
&. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .
\end{aligned}$$

Putting these values in (a), we have the double series

$$\begin{aligned}
(c) \quad u = {} & u + (c_2 - a_2)u^2 + (c_3 - 2a_2c_2 - a_3)u^3 \\
& + [c_4 - a_2(c_2^2 + 2c_3) - 3a_3c_2 - a_4]u^4 \\
& + [c_5 - 2a_2(c_4 + c_2c_3) - 3a_3(c_2^2 + c_3) \\
& \qquad\qquad\qquad\qquad - 4a_4c_2 - a_5]u^5 \\
& + \ .
\end{aligned}$$

Equating coefficients of like powers of u on both sides, we get

$$\begin{aligned}
c_2 &= a_2, \\
c_3 &= 2a_2c_2 + a_3, \\
(d) \qquad c_4 &= a_2(c_2^2 + 2c_3) + 3a_3c_2 + a_4, \\
c_5 &= 2a_2(c_4 + c_2c_3) + 3a_3(c_2^2 + c_3) + 4a_4c_2 + a_5, \\
&. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ ,
\end{aligned}$$

from which we can determine all the coefficients c. The resulting series will formally satisfy (a). To complete the investigation, it is necessary to show that this series converges for some $u \neq 0$.

Since the series (a) converges for $x = 1$, we must have $a_n \to 0$, and we may write

$$|a_n| < \text{some } \alpha \ (n = 2, 3, \ . \ . \ .).$$

From (d) we see that c_n is a polynomial in $a_2, a_3, a_4, \ . \ . \ . \ ,$ a_n with positive integral coefficients,

$$c_n = f_n(a_2, a_3 \ . \ . \ . \ , a_n).$$

If $\gamma_n = f_n(\alpha, \ \alpha, \ . \ . \ . \ , \ \alpha)$, we have $|c_n| < \gamma_n$. If we replace all the a's in (a) by α, we get the geometric series

$$\begin{aligned} u &= x - \alpha x^2 - \alpha x^3 - \alpha x^4 - \ . \ . \ . \\ (e) \qquad &= x - \frac{\alpha x^2}{1 - x}. \end{aligned}$$

The inverse series belonging to (e) is

$$(f) \qquad x = u + \gamma_2 u^2 + \gamma_3 u^3 + \gamma_4 u^4 + \ . \ .$$

Solving (e) for x, we obtain

$$(g) \qquad x = \frac{1 + u + \sqrt{1 - 2(2\alpha + 1)u + u^2}}{2(1 + \alpha)}.$$

Putting $1 - 2(2\alpha + 1)u + u^2 = 1 - v$, for u near $u = 0$, we have $|v| < 1$, and by the binomial theorem,

$$\sqrt{1 - v} = 1 + d_1 v + d_2 v^2 + \ . \ . \ .$$

Replacing v by its value in u, this becomes a power series in u which converges for u near the origin, by Art. **239.** Hence we see from (g) that x can be developed in a power series about the origin, and therefore series (f) converges about $u = 0$, and then (b) converges in some interval about $u = 0$.

We have then the

Theorem.—*If the series*

$$(h)\qquad u = b + b_1x + b_2x^2 + b_3x^3 + \cdots, \quad (b_1 \neq 0),$$

converges about the point $x = 0$, x *is defined as a function of* u *which admits the development*

$$(k)\quad x = (u - b)\left\{\frac{1}{b_1} + a_1(u - b) + a_2(u - b)^2 + \cdots\right\}$$

about the point $u = b$. *The coefficients* a *may be obtained by substituting* (k) *in* (h), *and using the method of undetermined coefficients.*

This process is called **reversion of series,** or sometimes **inversion of series.**

Example.—Starting from

$$u = \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

and putting

$$x = u + c_2u^2 + c_3u^3 + \cdots,$$

we find

$$\begin{aligned}
-c_2 &= -\tfrac{1}{2}, \therefore c_2 = \tfrac{1}{2},\\
-c_3 &= 2(-\tfrac{1}{2})(\tfrac{1}{2}) + \tfrac{1}{3} = -\tfrac{1}{6}, \therefore c_3 = \tfrac{1}{6},\\
-c_4 &= -\tfrac{1}{2}(\tfrac{1}{4} + 2\cdot\tfrac{1}{6}) + 3(\tfrac{1}{3})(\tfrac{1}{2} - \tfrac{1}{4}) = -\tfrac{1}{24},\\
&\qquad\qquad \therefore c_4 = \tfrac{1}{24},\\
-c_5 &= 2(-\tfrac{1}{2})(\tfrac{1}{24} + \tfrac{1}{2}\cdot\tfrac{1}{6}) + 3(\tfrac{1}{3})(\tfrac{1}{4} + \tfrac{1}{6})\\
&\qquad + 4(-\tfrac{1}{4})(\tfrac{1}{2}) + \tfrac{1}{5} = -\tfrac{1}{120}, \therefore c_5 = \tfrac{1}{120},\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Hence

$$x = u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24} + \frac{u^5}{120} + \cdots,$$

or

$$1 + x = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + \cdots = e^u.$$

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CHAPTER XVII

INFINITE PRODUCTS

243. Convergence of Infinite Product.—Let u_1, u_2, u_3, . . . , u_n, . . . be an infinite sequence of real or complex numbers and put

$$P_n = (1 + u_1)(1 + u_2) \ . \ . \ . \ (1 + u_n) = \prod_{i=1}^{n} (1 + u_i).$$

When $n \rightarrow \infty$, P_n becomes an **infinite product,** and it may be denoted by $\prod\limits_{i=1}^{\infty}(1 + u_i)$.

The infinite product $\prod\limits_{1}^{\infty}(1 + u_n)$ *is called* **convergent** *when the sequence* (P_n) *is convergent, i.e., when* LP_n *exists, and when this limit* P *of the sequence is* **not zero.** *The limit* P *is called the* **value** *of the infinite product.*

The infinite product is said to be **divergent** *when* $P_n \rightarrow \infty$ *or* $P_n \rightarrow 0$, *and* **oscillating** *when* P_n *does not approach any definite limit.*

The reason for excluding products whose limit P is zero from the class of convergent products is that a product of a finite number of factors cannot vanish unless one at least of its factors itself vanishes, and this general principle would be violated by zero products, hence they are classed among non-convergent products. As an example of a zero-product, we may take

$$P = 1 \cdot \tfrac{1}{2} \cdot \tfrac{2}{3} \cdot \tfrac{3}{4} \cdot \tfrac{4}{5} \ . \ . \ . \ ,$$

for $P_n = \dfrac{1}{n} \rightarrow 0$. To simplify the following theory, we shall assume that no $u_n = -1$, so that $1 + u_n \neq 0$ for every n.

If the infinite product $\prod(1+u_n)$ is convergent, then by the general condition of convergence of a sequence,

$$|P_{n+p} - P_n| < \epsilon,\ n > N,\ p = 1, 2, 3, \ . \ . \ . \ ,$$

and also we must have

$$|P_n| > g$$

for every n, where g is a positive constant.

244. A General Condition for Convergence. Theorem.—*The necessary and sufficient condition for convergence of the infinite product* $\prod\limits_1^\infty(1 + u_n)$ *is that, for any arbitrarily small positive* ϵ, *an index* N *can be found such that for* $n \geqq N$,

$$\text{(a)} \qquad \left|\frac{P_{n+p}}{P_n} - 1\right| < \epsilon,$$

where p *may have any positive integral value.*

Suppose first that the product is convergent, then since $|P_n| > g$, we can divide both sides of

$$|P_{n+p} - P_n| < \epsilon$$

by $|P_n|$, and we get

$$\left|\frac{P_{n+p}}{P_n} - 1\right| < \frac{\epsilon}{g}.$$

The condition is therefore necessary.

If now the condition (*a*) is satisfied, for any assigned ϵ, let us determine the corresponding n; then for this n, $|P_n| = A$ is a positive finite number $\neq 0$. Multiply both sides of (*a*) by $|P_n| = A$, and we get

$$|P_{n+p} - P_n| < A\epsilon,$$

or

$$A(1 - \epsilon) < |P_{n+p}| < A(1 + \epsilon).$$

The first of these inequalities shows that LP_n exists, and the second shows that this limit is not zero. Thus the condition is sufficient.

The inequality (a) can be written

$$(b) \qquad |(1 + u_{n+1})(1 + u_{n+2}) \; . \; . \; . \; (1 + u_{n+p}) - 1| < \epsilon$$

for $n > N$, and for every p.

245. Necessary Condition for Convergence.—If we put $p = 1$, we have

$$\frac{P_{n+p}}{P_n} = 1 + u_{n+1},$$

and Art. **244** gives

$$|u_{n+1}| < \epsilon, \; n > N,$$

and we have the

Theorem.—*The necessary condition for convergence of the infinite product* $\prod_1^\infty (1 + u_n)$ *is* $Lu_n = 0$.

This condition is *not* in general a *sufficient* condition. This is shown by the example of $u_n = \pm \frac{1}{n}$, for,

$$\prod_{i=1}^{n} \left(1 + \frac{1}{i}\right) = \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \; . \; . \; . \; \left(1 + \frac{1}{n}\right)$$

$$= \cancel{2} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{4}}{\cancel{3}} \cdot \frac{\cancel{5}}{\cancel{4}} \; . \; . \; . \; \frac{n+1}{\cancel{n}} = n + 1,$$

and

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i}\right) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \; . \; . \; . \; \left(1 - \frac{1}{n}\right)$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \frac{\cancel{4}}{\cancel{5}} \; . \; . \; . \; \frac{n-1}{n} = \frac{1}{n},$$

so that both infinite products are divergent, since for the first $P_n \to \infty$, and for the second $P_n \to 0$, and yet $Lu_n = 0$ for both.

246. Equivalent Series.—From the recursion formula

$$P_i = (1 + u_i)\, P_{i-1} = P_{i-1} + u_i P_{i-1},$$

we get

$$\begin{aligned} P_n &= P_{n-1} + u_n P_{n-1} \\ &= P_{n-2} + u_{n-1} P_{n-2} + u_n P_{n-1} \\ &= P_{n-3} + u_{n-2} P_{n-3} + u_{n-1} P_{n-2} + u_n P_{n-1}, \end{aligned}$$

and so on.

$$(a) \qquad \therefore P_n = 1 + \sum_{i=1}^{n} u_i P_{i-1}.$$

If the infinite product converges, then LP_n exists, and hence the series $1 + \sum_{1}^{\infty} u_i P_{i-1}$ is convergent with sum equal to the value of the product; and conversely, if the series is convergent, then LP_n exists and the product is convergent with value equal to the sum of the series:

$$(b) \qquad \prod_{1}^{\infty} (1 + u_n) = 1 + \sum_{1}^{\infty} u_n P_{n-1}.$$

This series $1 + \sum_{1}^{\infty} u_n P_{n-1}$ *is called*[1] *the series* **equivalent** *to the infinite product.*

By means of the formula (*b*), we can transform an infinite product into a series. Conversely, we can transform the partial sum U_n of a series:

$$U_n = u_1 + u_2 + \; . \; . \; . \; + u_n$$

into a product. For from the identity

$$1 + \frac{u_i}{U_{i-1}} = \frac{U_i}{U_{i-1}},$$

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[1] After Mittag-Leffler.

we [illegible]

$$(c) \qquad U_n = u_1 \cdot \prod_{i=1}^{n} \left(1 + \frac{u_i}{U_{i-1}}\right).$$

These two formulae, (*b*) and (*c*), are not in general practically applicable, since we can in general neither express the partial products P_n nor the partial sums U_n in simple form; but they may be of theoretical value.

247. Weierstrass' Inequalities. Theorem.—*If* $a_1, a_2, \ldots, a_n, \ldots$ *are all positive and* < 1, *then*

$$(a) \quad (1 + a_1)(1 + a_2) \ldots (1 + a_n) > 1 + (a_1 + a_2 + \ldots + a_n),$$

$$(b) \quad (1 - a_1)(1 - a_2) \ldots (1 - a_n) > 1 - (a_1 + a_2 + \ldots + a_n),$$

and if $\sum_{i=1}^{n} a_i < 1$,

$$(c) \quad (1 + a_1)(1 + a_2) \ldots (1 + a_n) < [1 - (a_1 + a_2 + \ldots + a_n)]^{-1},$$

$$(d) \quad (1 - a_1)(1 - a_2) \ldots (1 - a_n) < [1 + (a_1 + a_2 + \ldots + a_n)]^{-1}.$$

We have

$$(1 + a_1)(1 + a_2) = 1 + a_1 + a_2 + a_1a_2 > 1 + a_1 + a_2,$$

therefore

$$(1 + a_1)(1 + a_2)(1 + a_3) > (1 + a_1 + a_2)(1 + a_3) > 1 + a_1 + a_2 + a_3,$$

and so on. Similarly,

$$(1 - a_1)(1 - a_2) = 1 - (a_1 + a_2) + a_1a_2 > 1 - (a_1 + a_2),$$

and since $(1 - a_3)$ is positive,

$$(1 - a_1)(1 - a_2)(1 - a_3) > [1 - (a_1 + a_2)](1 - a_3) > 1 - (a_1 + a_2 + a_3),$$

and so on.

Now

$$1 + a_i = \frac{1 - a_i^2}{1 - a_i} < \frac{1}{1 - a_i},$$

hence

$$(1 + a_1)(1 + a_2) \ . \ . \ . \ (1 + a_n) < \frac{1}{(1 - a_1)(1 - a_2) \ . \ . \ . \ (1 - a_n)},$$

and since $\sum_1^n a_i < 1$, using (*b*), we get (*c*). Similarly we obtain (*d*).

Combining these results, we may write

$$(e) \qquad (1 - \sum a_i)^{-1} > \prod (1 + a_i) > 1 + \sum a_i,$$

$$(f) \qquad (1 + \sum a_i)^{-1} > \prod (1 - a_i) > 1 - \sum a_i.$$

248. Theorem.—*If* $a_1, a_2, \ . \ . \ . \ , a_n, \ . \ . \ .$ *all lie between* 0 *and* 1, *then the necessary and sufficient condition for the convergence of the infinite products* $\prod_1^\infty (1 + a_n)$ *and* $\prod_1^\infty (1 - a_n)$ *is the convergence of the series* $\sum_1^\infty a_n$.

Put

$$P_n = (1 + a_1)(1 + a_2) \ . \ . \ . \ (1 + a_n),$$
$$Q_n = (1 - a_1)(1 - a_2) \ . \ . \ . \ (1 - a_n).$$

Then it is evident that the sequence (P_n) is increasing and the sequence (Q_n) is decreasing.

Suppose first that $\sum a_n$ is convergent. Then we can find m so that if

$$\sigma_m = a_{m+1} + a_{m+2} + \ . \ . \ . \ ,$$

we have $\sigma_m < 1$. Then

$$\frac{1}{1-\sigma_m} > \frac{1}{1-(a_{m+1} + \ . \ . \ . \ + a_n)}.$$

But

$$\frac{P_n}{P_m} = (1 + a_{m+1})(1 + a_{m+2}) \ . \ . \ . \ (1 + a_n),$$

and applying inequality (*e*) of Art. **247**, we have

$$\frac{P_n}{P_m} < \frac{1}{1-(a_{m+1} + \ . \ . \ . + a_n)},$$

and therefore

$$\frac{P_n}{P_m} < \frac{1}{1-\sigma_m},$$

or

$$(a) \qquad P_n < \frac{P_m}{1-\sigma_m} \qquad (n > m).$$

Similarly, we find

$$\frac{Q_n}{Q_m} > 1 - (a_{m+1} + \ . \ . \ . \ + a_n) > 1 - \sigma_m,$$

or

$$(b) \qquad Q_n > Q_m(1-\sigma_m).$$

Now (P_n) is increasing and (*a*) shows that it is bounded above, and (Q_n) is decreasing and is bounded below by (*b*), hence both these sequences have limits (Art. **28**),

$$P \leqq \frac{P_m}{1-\sigma_m}, \ Q \leqq Q_m(1-\sigma_m).$$

Next suppose $\sum a_n$ divergent, then we can find m such that, for any positive number G, however large,

$$a_1 + a_2 + \ . \ . \ . \ + a_n > G, \ n > m.$$

By inequalities (*a*) and (*f*) of Art. **247**, we now get

$$P_n > 1 + G,\ Q_n < \frac{1}{1+G},\ n > m,$$

hence

$$LP_n = \infty,\ LQ_n = 0,$$

so that both products are divergent.

Examples.—(1) The infinite products $\prod\limits_1^\infty \left(1 + \frac{1}{n}\right)$ and $\prod\limits_2^\infty \left(1 - \frac{1}{n}\right)$ are divergent, since $\sum \frac{1}{n}$ is divergent.

(2) The infinite product $\prod\limits_2^\infty \left(1 - \frac{1}{n^2}\right)$ is convergent, since $\sum \frac{1}{n^2}$ is convergent. This may be verified directly, since

$$1 - \frac{1}{n^2} = \frac{(n-1)(n+1)}{n^2},$$

and we may write

$$Q_{n-1} = \frac{1 \cdot 3}{2^2} \cdot \frac{2 \cdot 4}{3^2} \cdot \frac{3 \cdot 5}{4^2} \cdot \frac{4 \cdot 6}{5^2} \quad . \quad . \quad . \quad \frac{(n-1)(n+1)}{n^2}$$
$$= \frac{1}{2}\,\frac{n+1}{n},$$

so that $LQ_{n-1} = \frac{1}{2}$. Similarly, $\prod\limits_1^\infty \left(1 + \frac{1}{n^2}\right)$ is convergent.

This theorem may also be proved by use of the equivalent series. Suppose first that $\sum a_n$ is convergent, then when n increases, P_n can only increase, and to prove that $P_n \to$ a limit we need only prove that P_n remains $<$ a fixed number. Developing the product, we have

$$P_n = (1 + a_1)(1 + a_2) \ . \ . \ . \ (1 + a_n)$$
$$= 1 + a_1 + a_2 + \ . \ . \ . \ + a_n + a_1a_2 + \ . \ . \ .$$
$$+ a_1a_2 \ . \ . \ . \ a_n = 1 + \sum_1^n a_iP_{i-1}.$$

Putting $A_n = a_1 + a_2 + \ . \ . \ . \ + a_n$, and $A = LA_n = \sum\limits_1^\infty a_i$, we have

$$P_n < 1 + \frac{A_n}{1!} + \frac{A_n{}^2}{2!} + \ . \ . \ . \ + \frac{A_n{}^n}{n!} < e^{A_n} < e^A,$$

hence $P_n \rightarrow$ a limit and the product is convergent. Conversely, if the product is convergent, P_n remains finite, and $1 + \sum a_iP_{i-1}$ can only be convergent when the series $\sum a_i$ is convergent, by Arts. **125, 127.**

249. Theorem.—*If* $\prod(1 + |u_n|)$ *is convergent, so also is* $\prod(1 + u_n)$.

If, in a polynomial in $u_1, u_2 \ . \ . \ . \ , u_n$, with positive coefficients, we replace $u_1, u_2, \ . \ . \ . \ , u_n$ by their absolute values, we obtain the sum of the absolute values of the terms of this polynomial is $\geqq$ the absolute value of the polynomial. But the polynomial $1 + u_1 + \ . \ . \ . \ + u_n + u_1u_2 + \ . \ . \ . \ + u_1u_2 \ . \ . \ . \ u_n$, which we obtain by effecting the development of the partial product P_n, has its coefficients positive and if m is any integer $> n$, we see that the development of P_m contains, besides the terms of P_n, other terms with positive coefficients. It follows that, if we regard $P_m - P_n$ as a polynomial developed and reduced, the coefficients of this polynomial will be positive, and hence if we denote by P'_n the result obtained by replacing in P_n the terms $u_1, u_2 \ . \ . \ . \ , u_n$ by their absolute values, we have

$$P'_n \geqq |P_n|, \ P'_m - P'_n \geqq |P_m - P_n|, \ (m > n).$$

Evidently $P'_m - P'_n \geqq 0$. Now since $\prod(1 + |u_n|)$ is convergent, $P'_n \to$ a limit as $n \to \infty$, and

$$\therefore P'_m - P'_n < \epsilon,\ m > n > \nu,$$

hence

$$|P_m - P_n| < \epsilon,\ m > n > \nu,$$

therefore $P_n \to$ a limit. We must next show that this limit is $\neq 0$.

Suppose first that, for every n, $|u_n| < 1$. We have

$$(a)\ \frac{1}{P_n} = \left(1 - \frac{u_1}{1 + u_1}\right)\left(1 - \frac{u_2}{1 + u_2}\right) \cdot \cdot \cdot \left(1 - \frac{1}{1 + u_n}\right).$$

Since $\prod(1 + |u_n|)$ is convergent, it follows from Art. **248** that $\sum|u_n|$ is convergent, then $\sum\left|\frac{u_n}{1 + u_n}\right|$ is convergent, since the ratio of the n^{th} terms of these two series is $|1 + u_n|$, which $\to 1$ as $n \to \infty$ (Art. **96**), and then by Art. **248** again we see that $\frac{1}{P_n} \to$ a finite limit, hence $P_n \to$ a limit $\neq 0$. We may now remove the restriction on u_n, *i.e.*, $|u_n| < 1$, by writing

$$P_m = (1 + u_1) \ . \ . \ . \ (1 + u_n) \times (1 + u_{n+1}) \ . \ . \ . \ (1 + u_m),\ m > n,$$

and

$$P = (1 + u_1) \ . \ . \ . \ (1 + u_n) \cdot \prod_{i=1}^{\infty} (1 + u_{n+i}),$$

and since $u_i \to 0$, we can take n so large that for every i, $|u_{n+i}| < 1$. Then by the preceding result, $\prod_{i=1}^{\infty}(1 + u_{n+i})$ will have a value $\neq 0$, and hence P must be $\neq 0$.

250. Absolute Convergence.—The infinite product $\prod_1^\infty (1 + u_n)$ is called **absolutely convergent** when the product $\prod_1^\infty (1 + |u_n|)$ is convergent.

Theorem.—*The necessary and sufficient condition for the absolute convergence of* $\prod (1 + u_n)$ *is that* $\sum u_n$ *be absolutely convergent.*

This follows at once from Arts. **248, 249.**

251. Theorem.—*If the terms* a_n *are positive, the value of the infinite product* $\prod (1 + a_n)$ *is independent of the order of its factors.*

For let us suppose that we rearrange the terms $a_1, a_2, \ldots, a_n, \ldots$, and obtain the sequence $b_1, b_2, \ldots, b_n, \ldots$, and let $P = \prod_1^\infty (1 + a_n)$. The product of as many factors as we please among the numbers $1 + a_n$ or $1 + b_n$ is $< P$; then the infinite product $\prod (1 + b_n)$ is convergent and its value is at most $= P$. If we now choose arbitrarily a number $Q < P$, we can determine a positive integer p such that the product $(1 + a_1) \ldots (1 + a_p) > Q$; if the sequence $b_1, b_2, \ldots, b_r$ contains all the terms of the sequence $a_1, a_2, \ldots, a_p$, the product $(1 + b_1) \ldots (1 + b_r) > Q$; then the value of the infinite product $\prod (1 + b_n)$ is $> Q$: it is then $= P$.

252. Theorem.—*If* $\prod (1 + u_n)$ *is absolutely convergent, it is independent of the order of its factors.*

Suppose that (b_r) is a rearrangement of (u_n). The infinite product $\prod_1^\infty (1 + |b_n|)$ will be convergent and will have the value P'. Denote by Q'_n the product of its n first factors and by Q_n the product of the n first factors of $\prod(1 + b_n)$. Then $\prod(1 + b_n)$ is absolutely convergent since $\sum |b_n|$ is convergent. We wish to prove that $Q = P$, where $Q = LQ_n$.

Since $\prod_1^\infty(1 + |u_n|)$ and $\prod(1 + |b_n|)$ have the same value, we can make correspond to each positive ϵ a positive integer p such that

$$|Q'_\mu - P'_\nu| < \epsilon, \mu, \quad \nu > p.$$

But if we suppose that μ is sufficiently large that all the factors in P'_ν are contained in Q'_μ, we have, by the preceding remark,

$$|Q_\mu - P_\nu| < Q'_\mu - P'_\nu,$$

$$\therefore |Q_\mu - P_\nu| < \epsilon, \quad \mu, \nu > p.$$

If we let $\mu \to \infty$, and call $Q = LQ_n$, we have

$$|Q - P_\nu| \leqq \epsilon,$$

and letting $\nu \to \infty$,

$$|Q - P| \leqq \epsilon,$$

or

$$Q = P.$$

253. The Associate Logarithmic Series.—*The series* $\sum \log\ (1 + u_i)$ *is called the* **associate logarithmic series** *of the infinite product* $\prod(1 + u_i)$. By taking logarithms, we can reduce the theory of infinite products to the theory of infinite series.

Consider the case of a positive product $\prod (1 + a_n)$. Put

$$\lambda_n = \sum_1^n \log (1 + a_i), \text{ then } \log P_n = \lambda_n, \text{ and } P_n = e^{\lambda_n}.$$

If P is convergent, then $P_n \to$ a finite limit $\neq 0$, and hence $\lambda_n \to$ a limit and the series λ is convergent. If the series λ is convergent, then $\lambda_n \to$ a limit λ, and hence $P_n \to$ a limit $\neq 0$, and the infinite product is convergent. We have then the

Theorem. —*The positive infinite product P and the associate logarithmic series λ converge or diverge simultaneously. When convergent, $P = e^{\lambda}$, and $\lambda = \log P$.*

254. Theorem.—*The positive series $\sum a_n$ and $\sum \log (1 + a_n)$ converge or diverge simultaneously.*

For,

$$\underset{n}{L} \frac{\log (1 + a_n)}{a_n} = 1 \quad (\text{Art. } \mathbf{56}),$$

and we may apply Arts. **96, 97.**

The result of Art. **248** follows at once from this theorem.

255. Theorem.—*If the series $\sum u_n$ is conditionally convergent and the series $\sum u_n^2$ is absolutely convergent, then the infinite product $\prod(1 + u_n)$ is conditionally convergent; if $\sum u_n$ is divergent but $\sum |u_n|^2$ is convergent, the product diverges.*

For we have from the logarithmic series,

$$\log (1 + u_i) = u_i - K_i \cdot u_i^2,$$

where $K_i \to \frac{1}{2}$, hence for every positive integral value of p we have

$$\sum_{n+1}^{n+p} \log (1 + u_i) = \sum_{n+1}^{n+p} u_i - \sum_{n+1}^{n+p} K_i \cdot u_i^2,$$

from which our theorem follows without much difficulty, making use of the associate logarithmic series.

256. Uniform Convergence of an Infinite Product.—*The infinite product*

$$P(x) = \prod_{1}^{\infty} [1 + u_n(x)],$$

whose factors are functions of x, is said to be **uniformly convergent** *in a region R when the sequence $[P_n(x)]$, formed from the partial products*

$$P_n(x) = \prod_{1}^{n} [1 + u_i(x)],$$

is uniformly convergent in R to a value which is finite and $\neq 0$ (Art. **162**).

257. Theorem.—*The infinite product $\prod [1 + u_n(x)]$ is uniformly convergent in a region R when and only when the series*

$$\lambda(x) = \sum_{1}^{\infty} \log [1 + u_n(x)]$$

is uniformly convergent in R.

For if the associate logarithmic series $\sum \log [1 + u_n(x)]$ is uniformly convergent, and we put

$$\lambda(x) = \lambda_n(x) + \rho_n(x),$$

we have for $n > N$,

$$|\rho_n(x)| < \epsilon$$

for every x in R, and then

$$P(x) = e^{\lambda_n(x) + \rho_n(x)} = P_n(x) \cdot e^{\rho_n(x)},$$

$$P(x) - P_n(x) = P(x)[1 - e^{-\rho_n(x)}].$$

Since $P(x)$ is finite, and since $e^{-\rho_n(x)} \to$ its limit 1 uniformly when $n \to \infty$, while x ranges over R, the product $\prod[1 + u_n(x)]$ must be uniformly convergent in R.

Conversely, if the product is uniformly convergent in R, and we put

$$P(x) = P_n(x)[1 + \sigma_n(x)],$$

we have for $n > N$, $|\sigma_n(x)| < \epsilon$ for every x in R, and we have

$$\lambda(x) = \lambda_n(x) + \log[1 + \sigma_n(x)] + 2q\pi i;$$

since $\log[1 + \sigma_n(x)] \to$ its limit 0 uniformly, the series $\sum \log[1 + u_n(x)]$ converges uniformly in R.

258. Theorem.—*If the series of positive terms* $\sum |u_n(x)|$ *is uniformly convergent in a region* R, *the infinite product* $\prod[1 + u_n(x)]$ *converges uniformly in* R.[1]

From the identity

$$\log[1 + u_n(x)] = u_n(x) \cdot k_n,$$

where k_n varies with x but so that the sequence $(k_n) \to 1$ while x ranges through R, we get

$$\left|\sum_{n+1}^{n+p} \log[1 + u_i(x)]\right| \leq G \cdot \sum_{n+1}^{n+p} |u_i(x),|$$

where G is the upper bound of the absolute values of the numbers (k_n), from which our theorem follows.

259. Theorem.—*If the infinite product* $\prod[1 + u_n(x)]$ *converges uniformly in a region* R, *and all the terms* $u_n(x)$ *are continuous functions of* x *in* R, *the value* $P(x)$ *of this product is a continuous function of* x *in* R.

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[1] Provided none of the functions $u_n(x)$ takes the value -1 as x ranges over R.

This follows at once from Art. **162.**

260. Theorem.—*If* $\sum M_n$ *is a convergent series of positive constants, and if for every* n *and for every* x *in a region* R, *we have* $|u_n(x)| \leq M_n$, *the infinite product* $\prod_1^\infty [1 + u_n(x)]$ *will be absolutely and uniformly convergent for every* x *in* R.

This theorem, which is the analogue of Weierstrass' M-test for series, may be easily proved by use of the equivalent series of the product.

261. Example.—Consider the infinite product

$$P(x) = \prod_1^\infty \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}.$$

We have

$$e^{-\frac{x}{n}} = 1 - \frac{x}{n} + \frac{x^2}{2n^2} - \frac{x^3}{6n^3} \cdot K_n,$$

where K_n remains finite when x has any finite value. Then we get

$$P(x) = \prod_1^\infty \left(1 - \frac{x^2}{2n^2} + \frac{x^3}{n^3} \cdot K'_n\right),$$

and we have

$$u_n(x) = -\frac{x^2}{2n^2} + \frac{x^3}{n^3} \cdot K'_n.$$

By Art. **258** we see that the product $P(x)$ is uniformly convergent for all finite x, which are not negative integers, and the function $P(x)$ is a continuous function for all these values of x. The function $P(x)$ evidently vanishes for negative integral values of x but for no other finite values of x.

This product is important in the theory of the Gamma function.

262. Example.—Consider the infinite product

$$\prod_{1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right),$$

which represents $\frac{\sin x}{x}$ (Art. **265**).

In order to find whether it is absolutely convergent, we must consider the series $\sum_{1}^{\infty}\frac{x^2}{n^2\pi^2}$, or $\frac{x^2}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}$; this series is absolutely convergent, and so the product is absolutely convergent for all finite values of x.

But now let this product be written in the form

$$\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\ \cdot\ \cdot\ \cdot$$

The absolute convergence of this product depends upon that of the series

$$-\frac{x}{\pi}+\frac{x}{\pi}-\frac{x}{2\pi}+\frac{x}{2\pi}-\ \cdot\ \cdot\ \cdot$$

But this series is only conditionally convergent, since its series of absolute values

$$\frac{|x|}{\pi}+\frac{|x|}{\pi}+\frac{|x|}{2\pi}+\frac{|x|}{2\pi}+\ \cdot\ \cdot\ \cdot$$

is divergent. In this form, therefore, the infinite product is not absolutely convergent, and if the order of its factors $\left(1\pm\frac{x}{n\pi}\right)$ is deranged, there is a risk of altering the value of the product. Lastly, let the same product be written in the form

$$\left\{\left(1-\frac{x}{\pi}\right)e^{\frac{x}{\pi}}\right\}\left\{\left(1+\frac{x}{\pi}\right)e^{-\frac{x}{\pi}}\right\}\left\{\left(1-\frac{x}{2\pi}\right)e^{\frac{x}{2\pi}}\right\}$$
$$\left\{\left(1+\frac{x}{2\pi}\right)e^{-\frac{x}{2\pi}}\right\}\ \cdot\ \cdot\ \cdot,$$

in which each of the expressions $\left(1 \pm \frac{x}{m\pi}\right)e^{\mp\frac{x}{m\pi}}$ is counted as a single element of the infinite product. The absolute convergence of this product depends on that of the series

$$\left\{\log\left(1-\frac{x}{\pi}\right)+\frac{x}{\pi}\right\}+\left\{\log\left(1+\frac{x}{\pi}\right)-\frac{x}{\pi}\right\}$$
$$+\left\{\log\left(1-\frac{x}{2\pi}\right)+\frac{x}{2\pi}\right\}+\ .\ .\ .\ ,$$

or

$$\left(-\frac{x^2}{2\pi^2}+\ .\ .\ .\right)+\left(-\frac{x^2}{2\pi^2}+\ .\ .\ .\right)+$$
$$\left(-\frac{x^2}{2\pi^2\cdot 2^2}+\ .\ .\ .\right)+\left(-\frac{x^2}{2\pi^2\cdot 2^2}+\ .\ .\ .\right)+\ .\ .\ .\ ,$$

and the absolute convergence of this series follows from that of the series

$$1+1+\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{4^2}+\ .\ .\ .$$

The infinite product in the last form is therefore again absolutely convergent, the introduction of the factors $e^{\pm\frac{x}{m\pi}}$ having changed the convergence from conditional to absolute.

263. Theorem.—*If* $\sum_1^\infty |a_n|$ *is convergent, the product* $\prod_1^\infty(1+a_nx)$ *is convergent for all values of* x, *and may be expressed as a power series* $P(x)$ *which is convergent for all finite values of* x.

The product $\prod(1+a_nx)$ is convergent, since $\sum |a_n|\,.\,|x|$ is convergent (Art. **250**). Let $|x| = r$, and suppose first that the coefficients a_1, a_2, a_3, . . . are all real and positive. and consider the partial product

$$P_n(r) = (1 + a_1r)(1 + a_2r) \quad . \quad . \quad . \quad (1 + a_nr).$$

We may expand this into

$$P_n(r) = 1 + s_1^{(n)}r + s_2^{(n)}r^2 + \quad . \quad . \quad . \quad + s_m^{(n)}r^m + \quad . \quad . \quad . \quad + s_n^{(n)}r^n,$$

where $s_1^{(n)}$ denotes the sum of the coefficients $a_1, a_2, \quad . \quad . \quad . \quad , a_n$; $s_2^{(n)}$ the sum of the products of these n numbers taken two at a time; $s_3^{(n)}$ the sum of the products taken three at a time; etc.

Now since any term $s_m^{(n)} \cdot r^m$ is $< P_n(r)$ and hence $< P(r)$, where $P(r) = \underset{n}{L}P_n(r)$, it follows that $\underset{n}{L}s_m^{(n)}r^m$ must exist, $= s_mr^m$ say. Then $\underset{n}{L}s_m^{(n)} = s_m$. But

$$P_n(r) > 1 + s_1^{(n)}r + s_2^{(n)}r^2 + \quad . \quad . \quad . \quad + s_m^{(n)}r^m,$$

and if we leave m fixed and let $n \to \infty$, we get

$$P(r) > 1 + s_1r + s_2r^2 + \quad . \quad . \quad . \quad + s_mr^m.$$

Then the series

$$1 + s_1r + s_2r^2 + \quad . \quad . \quad . \quad + s_nr^n + \quad . \quad . \quad .$$

has the sum $S_m(r)$ of its m first terms $< P(r)$, and hence $S_m(r) \to$ a limit $S(r)$ when $m \to \infty$, and $S(r) \leqq P(r)$. But since each of the terms of the product

$$P_n(r) = 1 + s_1^{(n)}r + s_2^{(n)}r^2 + \quad . \quad . \quad . \quad + s_n^{(n)}r^n$$

is less than the corresponding term of the sum

$$S_n(r) = 1 + s_1r + s_2r^2 + \quad . \quad . \quad . \quad + s_nr^n,$$

we have $S_n(r) > P_n(r)$, therefore we must have $S(r) \geqq P(r)$, and hence $S(r) = P(r)$.

Now consider the product

$$P_n(x) = (1 + a_1x)(1 + a_2x) \quad . \quad . \quad . \quad (1 + a_nx),$$

where the a_i are not necessarily real and positive, and $\sum_1 |a_n|$ is convergent. As before, we may write

$$P_n(x) = 1 + \sigma_1^{(n)}x + \sigma_2^{(n)}x^2 + \quad . \quad . \quad . \quad + \sigma_m^{(n)}x^m + \quad . \quad . \quad . \quad + \sigma_n^{(n)}x^n.$$

Any coefficient $\sigma_m^{(n)}$ is a sum of terms whose absolute values form expressions corresponding to $s_m^{(n)}$ above. Hence, when $n \rightarrow \infty$, since $s_m^{(n)} \rightarrow$ a limit s_m, the sum $\sigma_m^{(n)}$ must $\rightarrow$ a limit σ_m. The series

$$1 + \sigma_1 x + \sigma_2 x^2 + \ . \ . \ . + \sigma_n x^n + \ . \ . \ . = P(x) \text{ say,}$$

is convergent, since its terms are less in absolute value than the corresponding terms of $\sum s_n r^n$. The sum $S_n(x)$ contains all the terms of $P_n(x)$, and the terms of the difference $S_n(x) - P_n(x)$ have for their absolute values the corresponding terms of the difference $S_n(r) - P_n(r)$, and hence tend toward zero, as $n \rightarrow \infty$. It follows that $S_n(x) \rightarrow$ a limit $P(x)$.

Hence the function defined by the product $\prod(1 + a_n x)$ is developable into a uniformly convergent series $\sum \sigma_n x^n$ arranged according to increasing powers of x.

264. Double Infinite Products.—The discussion of double products is reducible to that of double series. The necessary and sufficient condition that $\prod(1 + u_{mn})$ may be absolutely convergent is that $\sum u_{mn}$ be absolutely convergent.

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CHAPTER XVIII

TRIGONOMETRIC DEVELOPMENTS. BERNOULLI AND EULER NUMBERS

265. Infinite Product for Sin x.—It is shown in advanced works on Trigonometry that when n is an odd integer,

$$\sin nx = a_0 \sin^n x + a_1 \sin^{n-1} x + \ . \ . \ . \ + a_{n-1} \sin x,$$

where the coefficients a_i are integers. If we put $\sin x = t$, we get

$$\sin nx = F_n(t) = a_0 t^n + a_1 t^{n-1} + \ . \ . \ . \ + a_{n-1} t.$$

Since $F_n(t)$ is a polynomial of degree n, by the fundamental theorem of Algebra, it must have n roots, which correspond to the values of x which make $\sin nx = 0$, namely,

$$0, \pm \sin \frac{\pi}{n}, \pm \sin \frac{2\pi}{n}, \ . \ . \ . \ , \pm \sin \frac{n-1}{2} \cdot \frac{\pi}{n}.$$

We may then write

$$F_n(t) = a_0 t \left(t - \sin \frac{\pi}{n}\right)\left(t + \sin \frac{\pi}{n}\right)\left(t - \sin \frac{2\pi}{n}\right) \ . \ . \ .$$

$$= a_0 t \left(t^2 - \sin^2 \frac{\pi}{n}\right)\left(t^2 - \sin^2 \frac{2\pi}{n}\right) \ . \ . \ . \ \left(t^2 - \sin^2 \frac{n-1}{2} \cdot \frac{\pi}{n}\right).$$

Dividing through by

$$\sin^2 \frac{\pi}{n} \cdot \sin^2 \frac{2\pi}{n} \ . \ . \ . \ \sin^2 \frac{n-1}{2} \cdot \frac{\pi}{n},$$

we get

$$\sin nx = K \sin x \left(1 - \frac{\sin^2 x}{\sin^2 \frac{\pi}{n}}\right)\left(1 - \frac{\sin^2 x}{\sin^2 \frac{2\pi}{n}}\right) \ . \ . \ .$$

$$\left(1-\frac{\sin^2 x}{\sin^2\frac{n-1}{2}\frac{\pi}{n}}\right),$$

where K is a constant. Dividing this last equation by $\sin x$, and observing that when $x \to 0$, $\frac{\sin nx}{\sin x} \to n$, we find $K = n$. Replacing x by $\frac{x}{n}$, and putting in this value of K, we have

$$\sin x = n \cdot \sin\frac{x}{n} \cdot \prod_{r=1}^{\frac{n-1}{2}}\left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{r\pi}{n}}\right).$$

Put

$$P(x) = \prod_{r=1}^{\infty}\left(1-\frac{x^2}{r^2\pi^2}\right),\ P(x,n) = \prod_{r=1}^{\frac{n-1}{2}}\left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{r\pi}{n}}\right),$$

$$\lambda(x) = \log P(x) = \sum_{1}^{\infty}\log\left(1-\frac{x^2}{r^2\pi^2}\right),$$

$$\lambda(x,n) = \log P(x,n) = \sum_{r=1}^{\frac{n-1}{2}}\log\left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{r\pi}{n}}\right)$$

If we write

$$\lambda(x,n) = \lambda_m(x,n) + \bar{\lambda}_m(x,n),$$
$$\lambda(x) = \lambda_m(x) + \bar{\lambda}_m(x),$$

then

$$|\lambda(x,n) - \lambda(x)| \leqq |\lambda_m(x,n) - \lambda_m(x)| + |\bar{\lambda}_m(x,n)| + |\bar{\lambda}_m(x)|.$$

If $0 < x < \frac{\pi}{2}$, we have $\frac{x}{2} < \sin x < x$, hence if $r >$ some m,

$$\frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} > \frac{\frac{x^2}{4n^2}}{\frac{r^2\pi^2}{n^2}} = \frac{x^2}{4r^2\pi^2},$$

hence for m_1 such that $\frac{|x|}{m_1} < 1$, we have

$$\log\left(1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}}\right) < \log\left(1 - \frac{x^2}{4\pi^2 r^2}\right), \ r > m.$$

But the series

$$\sum_{r=m_1}^{\infty}\left(1 - \frac{x^2}{4\pi^2 r^2}\right)$$

is convergent, for

$$-\log(1-\alpha) < \alpha + K\alpha^2 \ (K \text{ constant}),$$

$$\therefore -\sum \log\left(1 - \frac{x^2}{4\pi^2 r^2}\right) < \sum \frac{x^2}{4\pi^2 r^2} + K \sum \left(\frac{x^2}{4\pi^2 r^2}\right)^2$$

$$< \frac{x^2}{4\pi^2} \sum \frac{1}{r^2} + K \cdot \frac{x^4}{16\pi^4} \sum \frac{1}{r^4}.$$

But $\sum \frac{1}{r^2}$ and $\sum \frac{1}{r^4}$ are convergent, hence $\sum\left(1 - \frac{x^2}{4\pi^2 r^2}\right)$ is convergent. Therefore, if m is sufficiently large,

$$|\lambda_m(x, n)| < \frac{\epsilon}{3}, \ |\bar{\lambda}_m(x)| < \frac{\epsilon}{3}.$$

If $n >$ some ν, we have

$$|\lambda_m(x, n) - \lambda_m(x)| < \frac{\epsilon}{3}.$$

Then we get

$$|\lambda(x, n) - \lambda(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,\ n > \nu,$$

so that

$$\underset{n}{L}\lambda(x, n) = \lambda(x).$$

It follows that

$$\underset{n}{L}P(x, n) = \underset{n}{L}e^{\lambda(x, n)} = e^{\lambda(x)} = P(x).$$

We have

$$n \sin\frac{x}{n} = x \cdot \frac{\sin\frac{x}{n}}{\frac{x}{n}} \longrightarrow x \text{ as } n \longrightarrow \infty,$$

hence we have finally

$$(a) \qquad \sin x = x \cdot \prod_{r=1}^{\infty}\left(1 - \frac{x^2}{r^2\pi^2}\right).$$

We have thus expressed sin x as an infinite product, which exhibits in simple form the zeros of the function.

An equivalent infinite product form for sin x is

$$(b) \qquad \sin x = x \cdot \prod_{n=-\infty}^{+\infty}\left(1 - \frac{x}{n\pi}\right)e^{\frac{x}{n\pi}}.$$

For this product has been shown to be convergent in Art. **262**, and if we group the factors in pairs, we have

$$\left(1 - \frac{x}{n\pi}\right)e^{\frac{x}{n\pi}}\left(1 + \frac{x}{n\pi}\right)e^{-\frac{x}{n\pi}} = \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

If we replace x by ix in the preceding product, we obtain

(c) $$\sinh x = x \cdot \prod_{1}^{\infty}\left(1 + \frac{x^2}{r^2\pi^2}\right).$$

266. Infinite Product for Cos x.—We may derive the infinite product representation of cos x from that for sin x by using the relation

$$\sin 2x = 2 \sin x \cos x.$$

We get

$$\cos x = \frac{1}{2}\cdot\frac{2x}{x}\cdot\frac{\prod\left(1 - \frac{4x^2}{n^2\pi^2}\right)}{\prod\left(1 - \frac{x^2}{n^2\pi^2}\right)}$$

$$= \frac{\prod\left(1 - \frac{4x^2}{(2m)^2\pi^2}\right)\cdot\left(1 - \frac{4x^2}{(2m-1)^2\pi^2}\right)}{\prod\left(1 - \frac{x^2}{n^2\pi^2}\right)}$$

$$= \frac{\prod\left(1 - \frac{x^2}{m^2\pi^2}\right)}{\prod\left(1 - \frac{x^2}{n^2\pi^2}\right)}\cdot\prod\left(1 - \frac{4x^2}{(2m-1)^2\pi^2}\right)$$

$$\therefore \cos x = \prod_{1}^{\infty}\left(1 - \frac{4x^2}{(2n-1)^2\pi^2}\right).$$

As in Art. **265,** we find also

$$\cos x = \prod_{-\infty}^{+\infty}\left(1 - \frac{2x}{(2n-1)\pi}\right)e^{\frac{2x}{(2n-1)\pi}}.$$

267. Wallis' Formula.—In the infinite product for sin x, put $x = \frac{\pi}{2}$, and we get

$$1 = \frac{\pi}{2}\cdot\prod\left(1 - \frac{1}{4r^2}\right) = \frac{\pi}{2}\cdot\prod\frac{(2r-1)(2r+1)}{2r\cdot 2r},$$

or

$$\frac{\pi}{2} = \prod \frac{2r \cdot 2r}{(2r-1)\cdot(2r+1)} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6. \quad . \quad . \quad .}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7. \quad . \quad . \quad .},$$

which is **Wallis' formula.**

268. Stirling's Formula.—Wallis' formula may be written

$$\frac{\pi}{2} = \underset{n}{L} \frac{2^2\cdot 4^2\cdot 6^2 \quad . \quad . \quad . \quad (2n)^2}{3^2\cdot 5^2\cdot 7^2 \quad . \quad . \quad . \quad (2n-1)^2}\cdot\frac{1}{2n+1},$$

and dividing by $\frac{\pi}{2}$, taking the square root, and using

$\underset{n}{L} \frac{n}{\frac{1}{2}(2n+1)} = 1$, we obtain

$$\underset{n}{L} \frac{2\cdot 4\cdot 6 \quad . \quad . \quad . \quad 2n}{3\cdot 5\cdot 7 \quad . \quad . \quad . \quad 2n-1}\cdot\frac{1}{\sqrt{n\pi}} = 1.$$

Multiplying numerator and denominator by

$$2 . 4 \cdot 6 \quad . \quad . \quad . \quad 2n = 2^n \cdot n!,$$

we have

$$(a) \qquad \underset{n}{L} \frac{(n!)^2 2^{2n}}{(2n)!\sqrt{n\pi}} = 1.$$

Now put

$$\phi(n) = \frac{n!}{n^{n+\frac{1}{2}}\sqrt{2\pi}},$$

then

$$\phi(2n) = \frac{(2n)!}{2^{2n+1}n^{2n+\frac{1}{2}}\sqrt{\pi}},$$

so that

$$\frac{[\phi(n)]^2}{\phi(2n)} = \frac{(n!)^2\cdot 2^{2n}}{(2n)!\sqrt{n\pi}},$$

or by (a),

(b) $$\underset{n}{L}\frac{[\phi(n)]^2}{\phi(2n)} = 1.$$

Now

$$\frac{\phi(n)}{\phi(n+1)} = \frac{n!}{(n+1)!}\cdot\frac{(n+1)^{n+3/2}}{n^{n+1/2}} = \left(1+\frac{1}{n}\right)^{n+1/2},$$

or

$$\log\frac{\phi(n)}{\phi(n+1)} = \left(n+\frac{1}{2}\right)\ \log\left(1+\frac{1}{n}\right).$$

By the logarithmic series, we have

$$\log\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \ldots,$$

so that

$$\begin{aligned}(n+1/2)\log\left(1+\frac{1}{n}\right) &= 1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \ldots,\\ &\qquad + \frac{1}{2n} - \frac{1}{4n^2} + \frac{1}{6n^3} - \ldots\\ &= 1 + \frac{1}{12n^2} - \left(\frac{1}{4}-\frac{1}{6}\right)\frac{1}{n^3} + \left(\frac{1}{5}-\frac{1}{8}\right)\frac{1}{n^4} - \ldots\end{aligned}$$

(c) $$= 1 + \frac{1}{12n^2} - \frac{2}{4\cdot 6}\,\frac{1}{n^3} + \frac{3}{5\cdot 8}\,\frac{1}{n^4} - \ldots$$

This may be written

(d) $$\left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right) = 1 + \left(\frac{1}{12}\cdot\frac{1}{n^2} - \frac{2}{4\cdot 6}\,\frac{1}{n^3}\right) + \left(\frac{3}{5\cdot 8}\,\frac{1}{n^4} - \frac{4}{6\cdot 10}\,\frac{1}{n^5}\right) + \ldots$$

(e) $$= \left(1 + \frac{1}{12}\,\frac{1}{n^2}\right) - \left(\frac{2}{4.6}\,\frac{1}{n^3} - \frac{3}{5.8}\,\frac{1}{n^4}\right) - \ldots$$

The general term (ν^{th}) of (c) is

$$\frac{\nu-1}{(\nu+1)\cdot 2\nu}\cdot\frac{1}{n^\nu},$$

so that all the differences in (*d*) and (*e*) are positive, since

$$\frac{\nu-1}{(\nu+1)\cdot 2\nu}\cdot\frac{1}{n^\nu}-\frac{\nu}{(\nu+2)2(\nu+1)}\cdot\frac{1}{n^{\nu+1}}$$
$$=\frac{n(\nu^2+\nu-2)-\nu^2}{2\nu(\nu+1)(\nu+2)n^{\nu+1}}>0.$$

Hence, we have

$$1<\left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right)<1+\frac{1}{12n^2},$$

or

$$1<\log\frac{\phi(n)}{\phi(n+1)}<1+\frac{1}{12n^2}.$$

Replace n by $n+1$, $n+2$, . . . , $2n-1$ successively, and we get

$$1<\log\frac{\phi(n+1)}{\phi(n+2)}<1+\frac{1}{12(n+1)^2}<1+\frac{1}{12n^2},$$
$$1<\log\frac{\phi(n+2)}{\phi(n+3)}<1+\frac{1}{12(n+2)^2}<1+\frac{1}{12n^2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$1<\log\frac{\phi(2n-1)}{\phi(2n)}<1+\frac{1}{12(2n-1)^2}<1+\frac{1}{12n^2}$$

Adding these inequalities, we find

$$n<\log\frac{\phi(n)}{\phi(2n)}<n+\frac{1}{12n},$$

or

$$1<e^{-n}\frac{\phi(n)}{\phi(2n)}<e^{\frac{1}{12n}}$$

$$(f)\qquad\therefore \underset{n}{L}\,e^{-n}\frac{\phi(n)}{\phi(2n)}=1.$$

Now

$$\underset{n}{L}\,\frac{e^n\phi(2n)}{\phi(n)}=\underset{n}{L}\,e^n\phi(n)\cdot\frac{\phi(2n)}{[\phi(n)]^2}=1,$$

or

$$\underset{n}{L} e^n \phi(n) = 1,$$

$$(g) \qquad \therefore \underset{n}{L} \frac{n!}{e^{-n} n^{n+\frac{1}{2}} \sqrt{2\pi}} = 1.$$

This formula is known as Stirling's formula. It may also be written, in the infinitary notation,

$$n! \sim e^{-n} \cdot n^{n+\frac{1}{2}} \sqrt{2\pi}.$$

269. Development of Tan x, Cot x in Infinite Series.— From the infinite products for sin x and cos x, we get by taking logarithms,

$$\log \sin x = \log x + \sum_1^\infty \log \left(1 - \frac{x^2}{n^2\pi^2}\right),$$

$$\log \cos x = \sum_1^\infty \log \left(1 - \frac{4x^2}{(2n-1)^2\pi^2}\right).$$

Differentiating,[1] we get

$$(a) \qquad \cot x = \frac{1}{x} + 2 \sum_1^\infty \frac{x}{x^2 - n^2\pi^2},$$

$$(b) \qquad \tan x = 2 \sum_1^\infty \frac{x}{\left(\frac{2n-1}{2}\right)^2 \pi^2 - x^2},$$

valid for all values of x for which cot x and tan x are defined.

The series for cot x may be written

$$\cot x = \frac{1}{x} + \sum_{-\infty}^{+\infty}{}' \left(\frac{1}{x - n\pi} + \frac{1}{n\pi}\right).$$

These formulae (a) and (b) exhibit the functions cot x, tan x as series of rational functions whose poles are the poles of the given functions. They are analogous to the representation of a fraction as a sum of partial fractions in Algebra.

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[1] This differentiation is legitimate since the series are uniformly convergent as is seen by applying the M-test.

270. Development of Sec x, Csc x in Infinite Series.—Since $\csc x = \tan \frac{x}{2} + \cot x$, we have

$$\csc x = 2\sum_1^\infty \frac{\frac{x}{2}}{\left(\frac{2n-1}{2}\right)^2 \pi^2 - \frac{x^2}{4}} + \frac{1}{x} - 2\sum_1^\infty \frac{x}{n^2\pi^2 - x^2}$$

$$= \frac{1}{x} + \sum \frac{4x}{(2n-1)^2\pi^2 - x^2} - 2\sum \frac{x}{n^2\pi^2 - x^2},$$

$$\therefore \csc x = \frac{1}{x} + \sum_1^\infty \frac{(-1)^{n-1}2x}{n^2\pi^2 - x^2},$$

valid for $x \neq \pm n\pi$.

The series for $\sec x$ may be found from this by using the relation $\sec x = \csc\left(\frac{\pi}{2} - x\right)$, and making a few transformations, giving the result

$$\sec x = \sum_1^\infty (-1)^{n-1} \cdot \frac{(2n-1)\pi}{\left(\frac{2n-1}{2}\right)^2 \pi^2 - x^2},$$

valid for every x such that $\cos x \neq 0$.

271. Bernoulli Numbers.—A remarkable set of numbers, called the **Bernoulli numbers,** may be obtained by the expansion of $\frac{x}{e^x - 1}$ into a power series.

From the exponential series for e^x, we get

$$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$$

By Art. **230** we may expand the reciprocal $\frac{x}{e^x - 1}$ in powers of x, provided $|x| < \rho$, where ρ is such that

$$\frac{\rho}{2!} + \frac{\rho^2}{3!} + \frac{\rho^3}{4!} + \ . \ . \ . \ \leqq 1,$$

which condition is satisfied if we take

$$\frac{\frac{\rho}{2}}{1 - \frac{\rho}{3}} = 1, \text{ or } \rho = \frac{6}{5} = 1.2.$$

We may then write

$$\frac{x}{e^x - 1} = 1 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \ . \ . \ . \ ,$$

if $|x| < 1.2$. Changing x into $-x$, we get

$$\frac{x}{1 - e^{-x}} = 1 - A_1x + A_2x^2 - A_3x^3 + A_4x^4 - \ . \ . \ . \ .$$

Subtracting the first of these equations from the second, we find the identity

$$x = -2A_1x - 2A_3x^3 - 2A_5x^5 - \ . \ . \ . \ ,$$

from which we obtain, by Art. **177,**

$$A_1 = -\tfrac{1}{2}, A_3 = 0, A_5 = 0, A_7 = 0, \ . \ . \ . \ , \qquad A_{2r-1} = 0, \ . \ . \ .$$

Then

$$\frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + A_2x^2 + A_4x^4 + A_6x^6 + \ . \ . \ .$$

$$\therefore x = (1 - e^{-x})(1 + \frac{x}{2} + A_2x^2 + A_4x^4 + \ . \ . \ . \ + A_{2n}x^{2n} + \ . \ . \ .),$$

$$= \left(\frac{x}{1} - \frac{x^2}{2!} + \frac{x^3}{3!} - \ . \ . \ . \ - \frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!} - \ . \ . \ .\right) \times \left(1 + \frac{x}{2} + A_2x^2 + A_4x^4 + \ . \ . \ . + A_{2n}x^{2n} + \ . \ . \ .\right).$$

These two factor series are convergent (absolutely), so we may multiply them, obtaining a new convergent series, in which all the coefficients, except that of x, must be zero. If we equate coefficients of odd powers of x, we find

$$\frac{A_{2n}}{1!} + \frac{A_{2n-2}}{3!} + \cdots + \frac{A_2}{(2n-1)!} - \frac{1}{2(2n)!} + \frac{1}{(2n+1)!} = 0,$$

or

$$(a) \quad \frac{A_{2n}}{1!} + \frac{A_{2n-2}}{3!} + \cdots + \frac{A_2}{(2n-1)!} = \frac{2n-1}{2(2n+1)!}.$$

If we equate coefficients of even powers of x, we find

$$(b) \qquad \frac{A_{2n}}{2!} + \frac{A_{2n-2}}{4!} + \cdots + \frac{A_2}{(2n)!} = \frac{2n}{2(2n+2)!}$$

Now put

$$(c) \qquad A_{2n} = (-1)^{n-1}\frac{1}{(2n)!} \cdot B_n;$$

then we have

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + B_1\frac{x^2}{2!} - B_2 \cdot \frac{x^4}{4!} + B_3 \cdot \frac{x^6}{6!} - \cdots$$

These coefficients B_1, B_2, B_3, . . . are called **Bernoulli's numbers.** They satisfy the relations (a) and (b). If we put $n = 1, 2, 3, \ldots$, successively, in equation (a) or (b), we can calculate, one after the other, these numbers B_1, B_2, B_3, ... We thus find for the first few the values:

$$B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, B_4 = \frac{1}{30}, B_5 = \frac{5}{66}, B_6 = \frac{691}{2730},$$
$$B_7 = \frac{7}{6}, B_8 = \frac{3617}{510}, B_9 = \frac{43867}{798}, B_{10} = \frac{174611}{330}, \text{ etc.}$$

The following formal method of obtaining the B_r is worthy of notice. Assume the development

$$\frac{x}{e^x - 1} = 1 + A_1 x + \frac{A_2}{2!}x^2 + \frac{A_3}{3!}x^3 + \ . \ . \ .$$

Write the subscript in A_i as an exponent, formally, then

$$\frac{x}{e^x - 1} = 1 + \frac{Ax}{1!} + \frac{A^2x^2}{2!} + \frac{A^3x^3}{3!} + \ . \ . \ . \ = e^{Ax}\,(\text{formally}).$$

$$\therefore\ x = e^{Ax}(e^x - 1) = e^{(A+1)x} - e^{Ax},$$

or

$$x = x + [(A + 1)^2 - A^2]\frac{x^2}{2!} + \left[(A + 1)^3 - A^3\right]\frac{x^3}{3!} + \ . \ . \ .$$

Equating coefficients of like powers, we get

$$(A + 1)^2 - A^2 = 0,\ (A + 1)^3 - A^3 = 0,\ .\ .\ .\ ,\ (A + 1)^r - A^r = 0,\ .\ .\ .\ ,$$

from which

$$2A^1 + 1 = 0,\ 3A^2 + 3A^1 + 1 = 0,\ 4A^3 + 6A^2 + 4A^1 + 1 = 0,\ .\ .\ .\ ,$$

or interpreting the exponents of A as subscripts,

$$2A_1 + 1 = 0,\ 3A_2 + 3A_1 + 1 = 0,\ 4A_3 + 6A_2 + 4A_1 + 1 = 0,\ .\ .\ .\ ,$$

from which we solve for A_1, A_2, A_3, . . . :

$$A_1 = -\tfrac{1}{2},\ A_2 = \tfrac{1}{6},\ A_3 = 0,\ A_4 = -\tfrac{1}{30},\ .\ .\ .\ .$$

272. Some Properties of the Bernoulli Numbers.—In the series for cot x in Art. **269,** put $y = 2ix$, then since

$$\cot x = \frac{\cos x}{\sin x} = i \cdot \frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}} = i \cdot \frac{e^{2ix} + 1}{e^{2ix} - 1},$$

we get

$$\frac{e^y + 1}{e^y - 1} = \frac{2}{y} + \sum_1^\infty \frac{4y}{y^2 + 4n^2\pi^2},$$

and from this

$$\frac{1}{e^x - 1} = \frac{1}{2}\left(\frac{e^x + 1}{e^x - 1} - 1\right) = \frac{1}{x} - \frac{1}{2} + \sum_1^\infty \frac{2x}{x^2 + 4n^2\pi^2}.$$

Now if $|x| < 2\pi$, each fraction in the series on the right can be expanded in powers of x, giving

$$\frac{2x}{x^2 + 4n^2\pi^2} = \frac{x}{2n^2\pi^2}\left[1 - \frac{x^2}{4n^2\pi^2} + \frac{x^4}{16n^4\pi^4} - \quad . \quad . \quad . \quad \right].$$

Substituting this in the preceding series, we obtain a double series which is absolutely convergent, since the series of absolute values may be obtained by expanding the convergent series

$$\sum_1^\infty \frac{2|x|}{4n^2\pi^2 - |x|^2}.$$

Rearranging this double series according to powers of x, we get

$$\frac{1}{e^x - 1} = \frac{1}{x} - \frac{1}{2} + \frac{x}{2\pi^2}\left(\sum_1^\infty \frac{1}{n^2}\right) - \frac{x^3}{2^3\pi^4}\left(\sum_1^\infty \frac{1}{n^4}\right) + \frac{x^5}{2^5\pi^6}\left(\sum_1^\infty \frac{1}{n^6}\right) - \quad . \quad . \quad .$$

Comparing this expansion with the previous one for $\frac{x}{e^x - 1}$, we find

$$B_1 = \frac{1}{\pi^2}\sum_1^\infty \frac{1}{n^2}, \quad B_2 = \frac{3}{\pi^4}\sum_1^\infty \frac{1}{n^4}, \quad B_3 = \frac{45}{2\pi^6}\sum_1^\infty \frac{1}{n^6}, \quad . \quad . \quad . \quad ,$$

and generally

$$(a) \qquad B_r = \frac{2(2r)!}{(2\pi)^{2r}} \cdot \sum_1^\infty \frac{1}{n^{2r}}.$$

From the values of B_1, B_2, B_3 . . . given before, we obtain

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6}, \sum \frac{1}{n^4} = \frac{\pi^4}{90}, \sum \frac{1}{n^6} = \frac{\pi^6}{945}, \sum \frac{1}{n^8} = \frac{\pi^8}{9450}, \cdot \cdot$$

If, instead of using the expansion for cot x as above, we had used the expansion of tan x of Art. **269,** we should have obtained the formula

$$(b) \qquad B_r = \frac{2(2r)!}{(2\pi)^{2r}(1 - 2^{-2r})} \sum_1^\infty \frac{1}{(2n - 1)^{2r}}.$$

This formula may also be derived from (a) as follows. If we multiply the series $\sum \frac{1}{n^{2r}}$ by $1 - \frac{1}{2^{2r}}$, the effect is to deprive the series of all terms whose denominators are multiples of 2. Hence,

$$\left(1 - \frac{1}{2^{2r}}\right) \sum_1^\infty \frac{1}{n^{2r}} = \sum_1^\infty \frac{1}{(2n - 1)^{2r}}.$$

Using this formula and the numerical values of the B_r, we find

$$\sum_1^\infty \frac{1}{(2n - 1)^2} = \frac{\pi^2}{8}, \sum_1^\infty \frac{1}{(2n - 1)^4} = \frac{\pi^4}{96}, \sum_1^\infty \frac{1}{(2n - 1)^6} = \frac{\pi^6}{960}, \quad . \; . \; .$$

Formula (a) shows that the Bernoulli numbers are all positive. It can be shown[1] that the B_r increase steadily after B_3, and that $\underset{r}{L} B_r = \infty$.

Since

$$\underset{r}{L} \sum_1^\infty \frac{1}{n^{2r}} = 1,$$

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[1] See for instance, Chrystal, "Algebra," vol. II, p. 364.

and

$$B_r = \frac{2(2r)!}{(2\pi)^{2r}} \cdot \sum_1^\infty \frac{1}{n^{2r}},$$

and

$$n! \sim e^{-n} n^{n+\frac{1}{2}} \sqrt{2\pi},$$

we may represent the approximate value of B_r for large values of r by the formula

$$B_r = 4e^{-2r} r^{2r+\frac{1}{2}} \pi^{\frac{1}{2}-2r},$$

or

$$\log B_r = 2 \log 2 + \tfrac{1}{2} \log \pi + 2r(\log r - \log e - \log \pi) + \tfrac{1}{2} \log r.$$

273. Development of Log sin x, Tan x, etc. in Power Series.—From the infinite product for sin x, we obtain

$$\log \frac{\sin x}{x} = \sum_1^\infty \log\left(1 - \frac{x^2}{n^2\pi^2}\right), \ |x| < \pi.$$

Since

$$-\log\left(1 - \frac{x^2}{n^2\pi^2}\right) = \frac{x^2}{n^2\pi^2} + \frac{1}{2} \cdot \frac{x^4}{n^4\pi^4} + \frac{1}{3} \cdot \frac{x^6}{n^6\pi^6} + \ . \ . \ . \ ,$$

we get the double series

$$\begin{aligned} -\log \frac{\sin x}{x} = {} & \frac{x^2}{\pi^2} + \frac{1}{2}\frac{x^4}{\pi^4} + \frac{1}{3}\frac{x^6}{\pi^6} + \ . \ . \ . \\ & + \frac{x^2}{2^2\pi^2} + \frac{1}{2} \cdot \frac{x^4}{2^4\pi^4} + \frac{1}{3}\frac{x^6}{2^6\pi^6} + \ . \ . \ . \\ & + \frac{x^2}{3^2\pi^2} + \frac{1}{2} \cdot \frac{x^4}{3^4\pi^4} + \frac{1}{3}\frac{x^6}{3^6\pi^6} + \ . \ . \ . \\ & + \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \end{aligned}$$

if we sum by rows. Since the series is a positive series, we may sum by columns, and get

$$-\log\frac{\sin x}{x}=\frac{x^2}{\pi^2}\sum_1^\infty\frac{1}{n^2}+\frac{x^4}{2\pi^4}\sum_1^\infty\frac{1}{n^4}+\frac{x^6}{3\pi^6}\sum_1^\infty\frac{1}{n^6}$$
$$+\ \ldots,\ |x|<\pi.$$

Putting in the values of $\sum_1^\infty \frac{1}{n^{2r}}$ from Art. **272,** we find

$$-\log\frac{\sin x}{x}=B_1x^2+B_2\cdot\frac{2^3}{2\cdot 4!}x^4+B_3\cdot\frac{2^5}{3\cdot 6!}x^6$$
$$+B_4\cdot\frac{2^7}{4\cdot 8!}x^8+\ \ldots$$

Similarly we find

$$-\log\cos x=\frac{2^2x^2}{\pi^2}\sum_1^\infty\frac{1}{(2n-1)^2}+\frac{2^4x^4}{2\pi^4}\sum_1^\infty\frac{1}{(2n-1)^4}$$
$$+\frac{2^6x^6}{3\pi^6}\sum_1^\infty\frac{1}{(2n-1)^6}+\ \ldots,$$

or using the values of $\sum_1^\infty\frac{1}{(2n-1)^{2r}}$ from Art. **272,** we find

$$-\log\cos x=(2^2-1)\frac{2}{2!}B_1x^2+\frac{1}{2}(2^4-1)\frac{2^3}{4!}B_2x^4$$
$$+\frac{1}{3}(2^6-1)\frac{2^5}{6!}B_3x^6+\ \ldots$$

If we differentiate these power series for $-\log\frac{\sin x}{x}$ and $-\log\cos x$, we get

$$\tan x=\frac{2^2(2^2-1)}{2!}B_1x+\frac{2^4(2^4-1)}{4!}B_2x^3$$
$$+\frac{2^6(2^6-1)}{6!}B_3x^5+\ \ldots,|x|<\frac{\pi}{2},$$

and

$$\cot x=\frac{1}{x}-\frac{2^2}{2!}B_1x-\frac{2^4}{4!}B_2x^3-\frac{2^6}{6!}B_3x^5-\ \ldots,$$
$$0<|x|<\pi.$$

274. The Power Series for Sec x. The Euler Numbers. Since sec x is an even function: sec $(-x) =$ sec x, we may assume its expansion in the form

$$\sec x = a_0 + a_1x^2 + a_2x^4 + a_3x^6 + \cdots.$$

Multiply this series by

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

and equate coefficients of like powers of x on both sides of this result, and we find

$$a_0 = 1,\ a_r + \sum_{1}^{r} (-1)^i \frac{a_{r-i}}{(2i)!} = 0.$$

Now put

$$a_r = \frac{E_r}{(2r)!},$$

then we have

$$(a) \qquad \sum_{0}^{r-1} (-1)^i \binom{2r}{2i} E_{r-i} + (-1)^r = 0.$$

The numbers E_r are called **Euler's numbers.** They may be found from the above relation; for $r = 1, 2, 3, \ldots,$ we have $E_1 = 1$, $E_2 - 6E_1 + 1 = 0$, $E_3 - 15E_2 + 15E_1 - 1 = 0$, $E_4 - 28E_3 + 70E_2 - 28E_1 + 1 = 0$, etc., from which

$E_1 = 1,$
$E_2 = 5,$
$E_3 = 61,$
$E_4 = 1385,$
$E_5 = 50521,$

$E_6 = 270\ 2765,$
$E_7 = 19\ 936\ 0981,$
$E_8 = 1\ 939\ 151\ 2145,$
$E_9 = 240\ 487\ 967\ 5441,$
$\cdots\cdots\cdots$

A remarkable property of these numbers is that they end alternately in 1 and 5.

The sec x series is then

$$\sec x = 1 + \sum_{1}^{\infty} E_n \cdot \frac{x^{2n}}{(2n)!}.$$

REFERENCES

Pierpont, "Theory of Functions of Real Variables," vol. II, chap. VII.

Bromwich, "Introduction to the Theory of Infinite Series," arts. **70, 71, 91, 92, 93.**

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Stolz and Gmeiner, "Einleitung in die Funktionentheorie," vol. II, pp. 439–456.

Wilson, "Advanced Calculus," arts. **171–172, 170.**

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CHAPTER XIX

DIRICHLET SERIES

275. Dirichlet Series.—*A series of the form* $\sum_{n=1}^{\infty} \frac{a_n}{n^x}$, *where the* a_n *are independent of* x, *is called an* **ordinary Dirichlet series.** *A more general form is* $\sum_{n=1}^{\infty} a_n e^{-\lambda_n x}$, *where* (λ_n) *is a real increasing divergent sequence; it is called a* **Dirichlet series of type** λ_n; *sometimes the form* $\sum_{1}^{\infty} \frac{a_n}{\mu_n{}^x}$ *is used.*

If $\lambda_n = n$, the series is a power series in e^{-x}. If $\lambda_n = \log n$, then the series becomes the ordinary Dirichlet series $\sum a_n n^{-x}$.

The variable x may be real or complex, but we shall consider only real values.

276. Region of Convergence.—An ordinary Dirichlet series may converge for all values of x, or for no values of x, or may converge for some values and diverge for others. For example, the series $\sum_{1}^{\infty} \frac{1}{n!} \cdot \frac{1}{n^x}$ is convergent for every x, for by the ratio test, we have for every x,

$$\underset{n}{L}\left\{\frac{1}{(n+1)!(n+1)^x} : \frac{1}{n!\,n^x}\right\} = \underset{n}{L}\,\frac{1}{n+1} \cdot \frac{n^x}{(n+1)^x} = 0;$$

on the other hand, the series $\sum_{1}^{\infty} \frac{n!}{n^x}$ is always divergent, for by the ratio test,

$$\underset{n}{L}\frac{n!}{n^x}:\frac{(n+1)!}{(n+1)^x}=\underset{n}{L}\frac{1}{n+1}\cdot\frac{(n+1)^x}{n^x}=0$$

for every x.

Aside from these two extreme cases the region of convergence has the form given by the following

Theorem.—*If* $\sum_1^\infty \frac{a_n}{n^x}$ *is neither convergent everywhere nor nowhere, there exists a number* α *such that the series converges for* $x > \alpha$ *and diverges for* $x < \alpha$.

This number α is called the **convergence point**[1] of the Dirichlet series. When the series converges for all values of x, we may write $\alpha = -\infty$, and when it diverges for all values $\alpha = +\infty$.

Suppose that $\sum_1^\infty \frac{a_n}{n^x}$ converges for $x = x_0$ or oscillates (between finite bounds) for $x = x_0$. If we put

$$\sum_{n=1}^{p}\frac{a_n}{n^{x_0}} = S(p),$$

then for every p,

$$|S(p)| < A \text{ (const.)}.$$

Now if r, s are any integers: $s \geqq r \geqq 1$, and h is any positive number, we have

$$\sum_{n=r}^{s}\frac{a_n}{n^{x_0+h}} = \sum_{n=r}^{s}\frac{S(n)-S(n-1)}{n^h}$$

$$= \sum_{n=r}^{s} S(n)\cdot\left(\frac{1}{n^h}-\frac{1}{(n+1)^h}\right)-\frac{S(r-1)}{r^h}+\frac{S(s)}{(s+1)^h},$$

(applying Abel's identity, Art. **41**). Then

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[1] Also called the abscissa of convergence.

$$\left|\sum_{n=r}^{s} \frac{a_n}{n^{x_0+h}}\right| < A \cdot \sum_{n=r}^{s}\left(\frac{1}{n^h} - \frac{1}{(n+1)^h}\right) + \frac{A}{r^h} + \frac{A}{(s+1)^h} = \frac{2A}{r^h}$$

Now for arbitrarily small $\epsilon > 0$, we can choose r_0 so that for every $s \geqq r \geqq r_0$, we have

$$\left|\sum_{n=r}^{s} \frac{a_n}{n^{x_0+h}}\right| < \epsilon$$

for every h. Hence, the series $\sum_{1}^{\infty} \frac{a_n}{n^{x_0+h}}$ converges for all values of h, that is, $\sum_{1}^{\infty} \frac{a_n}{n^x}$ converges for every $x > x_0$.

We may now separate all real numbers into two classes: (1) those for which $\sum \frac{a_n}{n^x}$ diverges, and (2) those for which $\sum \frac{a_n}{n^x}$ converges. By the above proof, every number of the first class is < every number of the second class, and there are numbers in both classes. Therefore this cut defines a number α such that $\sum \frac{a_n}{n^x}$ diverges for $x < \alpha$ and converges for $x > \alpha$.

Thus *the interval of convergence of a real Dirichlet series is the part of the real axis to the right of some fixed point, the convergence-point.* It may be shown that when x is complex, the region of convergence is the half-plane to the right of the line given by the real part of α.

At the convergence-point $x = \alpha$, the series may converge or diverge. Thus, the series $\sum_{1}^{\infty} \frac{1}{n^x}$ diverges for $x \leqq 1$ and converges for $x > 1$, so that the series is divergent at the convergence-point $x = 1$; for this series is the hyperharmonic series. The series

$$\sum_{n=2}^{\infty} \frac{\dfrac{1}{\log^2 n}}{n^x} = \sum_{2}^{\infty} \frac{1}{n^x \log^2 n}$$

diverges for $x < 1$, for then

$$\sum_{n=2}^{s} \frac{1}{n^x \log^2 n} > \int_2^s \frac{du}{u^x \log^2 u} \geqq \frac{1}{\log^2 s} \int_2^s \frac{du}{u^x}$$
$$= \frac{1}{\log^2 s} \cdot \frac{1}{1-x} (s^{1-x} - 2^{1-x}),$$

and it converges for $x \geqq 1$, for then

$$\sum_{n=2}^{s} \frac{1}{n^x \log^2 n} \leqq \sum_{2}^{s} \frac{1}{n \log^2 n} \leqq \frac{1}{2 \log^2 2} + \int_2^s \frac{du}{u \log^2 u}$$
$$= \frac{1}{2 \log^2 2} + \left(\frac{1}{\log 2} - \frac{1}{\log s}\right) < \frac{1}{2 \log^2 2} + \frac{1}{\log 2};$$

so that this series converges for $x = 1$, its convergence-point.

The convergence-point α for the Dirichlet series $\sum a_n e^{-\lambda_n x}$ is given by the equation

$$\alpha = \overline{\underset{n}{L}} \frac{\log \left| \sum_{1}^{n} a_i \right|}{\lambda_n};$$

for proof of this result we must refer to Hardy and Riesz, "General Theory of Dirichlet's Series," pp. 6–8.

By applying the preceding results to the absolute value series $\sum |a_n| \cdot n^{-x}$, we obtain the following proposition on the absolute convergence of a Dirichlet series.

Theorem.—*For the Dirichlet series* $\sum a_n n^{-x}$ *there exists a number* $\bar{x}_0$ *such that the series is absolutely convergent for* $x > \bar{x}_0$ *and is not absolutely convergent for* $x < \bar{x}_0$.

It follows that every Dirichlet series has a convergence-point or abscissa and an interval or half-plane of absolute convergence. In general there will be an interval between the convergence-points of convergence and absolute convergence, although in special cases this interval may vanish or become infinite.

277. Uniform Convergence of a Dirichlet Series.—Theorem. *If a Dirichlet series*

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{n^x}$$

is convergent for $x = x_0$, then it is uniformly convergent for every $x \geqq x_0$.

For, if we put

$$R(p) = \sum_{n=p}^{\infty} \frac{a_n}{n^{x_0}},$$

then for every $h > 0$, and for every $p \geqq 2$,

$$\sum_{n=p}^{\infty} \frac{a_n}{n^{x_0+h}} = \sum_{n=p}^{\infty} \frac{R(n) - R(n+1)}{n^h}$$

$$= \sum_{p}^{\infty} R(n)\left(\frac{1}{n^h} - \frac{1}{(n-1)^h}\right) + \frac{R(p)}{(p-1)^h}.$$

Since $\sum_{1}^{\infty} \frac{a_n}{n^{x_0}}$ is convergent, for arbitrarily small $\epsilon > 0$, we have for every $n > n_0$, $|R(n)| < \frac{\epsilon}{2}$, hence for $p \geqq n_0$,

$$\left|\sum_{n=p}^{\infty} \frac{a_n}{n^{x_0+h}}\right| < \frac{\epsilon}{2} \cdot \sum_{n=p}^{\infty} \left(\frac{1}{(n-1)^h} - \frac{1}{n^h}\right) + \frac{\epsilon}{2(p-1)^h}$$

$$= \frac{\epsilon}{(p-1)^h} \leqq \epsilon,$$

for all values of $h > 0$, so that $\sum_{1}^{\infty} \frac{a_n}{n^x}$ is uniformly convergent for every $x \geqq x_0$.

From this theorem it follows that a Dirichlet series which converges for x_0 is continuous for every $x > x_0$ and is continuous at $x = x_0$ on the right, so that a Dirichlet series is continuous within its region of convergence.

278. Differentiation of a Dirichlet Series.—Theorem. *If* $f(x) = \sum_1^\infty \frac{a_n}{n^x}$ *has the convergence-point* α, *where* α *is finite or* $= -\infty$, *then* $f(x)$ *is differentiable for* $x > \alpha$ *and*

$$f'(x) = -\sum_1^\infty \frac{a_n \log n}{n^x}$$

Let x_0 be any value $>\alpha$, then $\sum_1^\infty \frac{a_n}{n^{x_0}}$ is convergent. Now the series $\sum_1^\infty \frac{a_n \log n}{n^x}$ may be obtained from $\sum_1^\infty \frac{a_n}{n^{x_0}}$ by multiplying its terms by the factor $\frac{\log n}{n^{x-x_0}}$; this factor decreases with increasing n, and the limit as $n \rightarrow \infty$ is 0, hence we may apply Art. **129,** showing that the series $\sum \frac{a_n \log n}{n^x}$ is convergent for every $x > x_0$ and therefore for every $x > \alpha$. Considered as a Dirichlet series, it is uniformly convergent for every $x > \gamma > \alpha$ (by Art. **275**), and since the derivative of $f(x)$ found by term-by-term differentiation is $-\sum_1^\infty \frac{a_n \log n}{n^x}$, the function $f(x)$ is differentiable for every $x > \alpha$, and our theorem follows.

279. The Zeta Function.—*The series* $\sum_1^\infty \frac{1}{n^x}$ *(the hyperharmonic series) defines a function, for every* $x > 1$, *which is called the Riemann* **Zeta function,** *denoted by* $\zeta(x)$. It is an important special case of Dirichlet series.

The series $\sum_{1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n^x}$ can be easily expressed in terms of the Zeta function. We have

$$\begin{aligned}\left(1-\frac{2}{2^x}\right)\zeta(x) &= \left(1-\frac{2}{2^x}\right)\left(1+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\ \cdot\ \cdot\ \cdot\ \right)\\ &= 1+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\frac{1}{5^x}+\frac{1}{6^x}+\ \cdot\ \cdot\ \cdot\\ &\qquad -\frac{2}{2^x}\qquad\quad -\frac{2}{4^x}\qquad\quad -\frac{2}{6^x}-\ \cdot\ \cdot\ \cdot\\ &= 1-\frac{1}{2^x}+\frac{1}{3^x}-\frac{1}{4^x}+\frac{1}{5^x}-\frac{1}{6^x}+\ \cdot\ \cdot\ \cdot\end{aligned}$$

$$\therefore \sum_{1}^{\infty}(-1)^{n+1}\frac{1}{n^x} = (1-2^{1-x})\,\zeta(x).$$

280. Multiplication of Dirichlet Series.—Let $\sum a_m e^{-\lambda_m x}$ and $\sum b_n e^{-\mu_n x}$ be any two general Dirichlet series, and let (ν_p) be the increasing sequence formed by all values of $\lambda_m + \mu_n$. Then if we put

$$c_p = \sum_{\lambda_m+\mu_n=\nu_p} a_m b_n,$$

the series $\sum c_p$ is called the **Dirichlet product** of the two series $\sum a_m$, $\sum b_n$ of type (λ, μ).

If $\lambda_m = \log m$, $\mu_n = \log n$, so that we are dealing with ordinary Dirichlet series, then $\nu_p = \log p$, and we have

$$c_p = \sum_{m\cdot n=p} a_m b_n = \sum_{d} a_d b_{p/d}$$

where the latter summation extends over all divisors d of p.

For Dirichlet multiplication of series there are theorems corresponding to those on Cauchy multiplication.[1]

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[1] See HARDY and RIESZ, "General Theory of Dirichlet Series."

Dirichlet series have very important applications in the Analytical Theory of Numbers.

REFERENCES

LANDAU, "Handbuch der Lehre von der Verteilung der Primzahlen," vol. I, chaps. VIII, IX.

HARDY, G. H., and RIESZ, M., "The General Theory of Dirichlet's Series."

NIELSEN, "Lehrbuch der unendlichen Reihen," chap. XXII.

KNOPP, "Unendliche Reihen," art. **58**.

CHAPTER XX

INFINITE INTEGRALS

When the range of integration of a definite integral is infinite, or when the integrand becomes infinite at some point of the range, the integral is called an **infinite integral.** We treat these two cases separately.

RANGE OF INTEGRATION INFINITE

281. Infinite Integral.—Let $f(x)$ be a function which is single-valued, finite and continuous for $x \geq a$, or discontinuous only at a finite number of points and integrable in this interval.

We define the infinite integral $\int_a^\infty f(x)dx$ *as equal to the limit* $\underset{x}{L} \int_a^x f(x)dx$ *when this limit exists, and we call the integral* **convergent.**

Similarly we define

$$\int_{-\infty}^{a} f(x)dx = \underset{x \to -\infty}{L} \int_{x}^{a} f(x)dx,$$

$$\int_{-\infty}^{+\infty} f(x)dx = \underset{x' \to -\infty}{L} \int_{x'}^{a} f(x)dx + \underset{x'' \to \infty}{L} \int_{a}^{x''} f(x)dx,$$

if the limits exist. In the latter case, if the limits do not exist separately, it may be possible that the limit

$$\underset{x \to \infty}{L} \int_{-x}^{x} f(x)dx$$

exists; we then call this limit the **principal value** of the integral.

Examples.—(1) $\int_1^\infty \frac{dx}{x^2} = \underset{x}{L} \int_1^x \frac{dx}{x^2} = \underset{x}{L}\left(1 - \frac{1}{x}\right) = 1.$

(2) $\int_0^\infty \frac{dx}{1 + x^2} = \underset{x}{L} \int_0^x \frac{dx}{1 + x^2} = \underset{x}{L}\left(\tan^{-1} x\right) = \frac{\pi}{2}.$

(3) $\int_0^\infty \frac{dx}{1 + x} = \underset{x}{L} \log (1 + x) = \infty$, (integral diverges).

282. General Condition for Convergence.—*The necessary and sufficient condition for the convergence of* $\int_a^\infty f(x)dx$ *is that, for any arbitrarily small* $\epsilon > 0$, *we can find a value* X *such that*

$$\left|\int_{x'}^{x''} f(x)dx\right| < \epsilon$$

for every $x'' \geqq x' \geqq X$.

This follows from the general principle of convergence of Art. **26.**

283. Absolute Convergence.—*The integral* $\int_a^\infty f(x)dx$ *will converge if the integral* $\int_a^\infty |f(x)|dx$ *converges*, since

$$\left|\int_{x'}^{x''} f(x)dx\right| \leqq \int_{x'}^{x''} \left|f(x)\right| dx.$$

The integral $\int_a^\infty f(x)dx$ *is said to be* **absolutely convergent** *when* $\int_a^\infty |f(x)|dx$ *converges; otherwise it is called* **conditionally convergent.**

284. Tests for Convergence.—If $f(x)$ is positive, the integral $\int_a^x f(x)dx$ evidently increases as x increases, hence by the principle of Art. **28,** we have the general test:

Theorem.—*If* $\int_a^x f(x)dx < A$ *for every* x, *where* A *is independent of* x, *the integral is convergent.*

From this we obtain at once the **comparison test** for convergence:

Theorem.—*If* $|f(x)| < g(x) > 0$, *and* $\int_a^\infty g(x)dx$ *is convergent, then* $\int_a^\infty f(x)dx$ *converges absolutely.*

For

$$\int_a^x |f(x)|dx < \int_a^x g(x)dx < \int_a^\infty g(x)dx = A.$$

Theorem.—*If* $x^k|f(x)| < A$ *(constant) for every* x, $(k > 1)$, *then* $\int_a^\infty f(x)dx$ *converges absolutely.*

For we have $|f(x)| < \dfrac{A}{x^k}$, and $\displaystyle\int_a^\infty \frac{dx}{x^k}$ is convergent when $k > 1$.

285. Uniform Convergence.—*The integral* $\int_a^\infty f(x,\alpha)dx$ *is said to* **converge uniformly** *with respect to* α *in an interval* (b, c) *if, corresponding to an arbitrarily small* $\epsilon > 0$, *a number* X *exists such that*

$$\left|\int_x^\infty f(x, \alpha)dx\right| < \epsilon$$

for every $x > X$ *and for all values of* α *in* (b, c).

This may be expressed in the form

$$\left|\int_{x'}^{x''} f(x, \alpha)dx\right| < \epsilon$$

for every $x'' > x' > X$ and for every α in (b, c).

On the basis of this definition of uniform convergence, the continuity, integration and differentiation of an infinite integral may be discussed in a manner similar to that for series.

INTEGRAND INFINITE

286. Improper Integrals.—*If* $f(x) \to \infty$ *when* $x \to b$, *we define the integral* $\int_a^b f(x)dx$ *by the limit*

$$\underset{\delta \to 0}{L} \int_a^{b-\delta} f(x)dx.$$

Similarly, when $f(x) \to \infty$ when $x \to a$, we define

$$\int_a^b f(x)dx = \underset{\delta \to 0}{L} \int_{a+\delta}^b f(x)dx;$$

and when $f(x) \to \infty$ when $x \to c$, where $a < c < b$, we define

$$\int_a^b f(x)dx = \underset{\delta \to 0}{L} \int_a^{c-\delta} f(x)dx + \underset{\delta' \to 0}{L} \int_{c+\delta'}^b f(x)dx,$$

or if these limits do not exist separately, by

$$\int_a^b f(x)dx = \underset{\delta \to 0}{L} \left\{ \int_a^{c-\delta} f(x)dx + \int_{c+\delta}^b f(x)dx \right\},$$

which is called the **principal value** of $\int_a^b f(x)dx$.

These infinite integrals are sometimes called **improper integrals.**

The properties of these integrals may be developed in a manner similar to the treatment of the case when the range of integration is infinite.

REFERENCES

Carslaw, "Fourier's Series and Integrals," chap. IV.
Bromwich, "Theory of Infinite Series," Appendix III.
Whittaker and Watson, "Modern Analysis," chap. IV.
Pierpont, "Theory of Functions of Real Variables," vol. I, chaps. XIV, XV.
Hobson, "Theory of Functions of a Real Variable," chap. V.
Wilson, "Advanced Calculus," chap. XIII.

CHAPTER XXI

FOURIER SERIES

287. Trigonometric Series.—*A series of the type*

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where the a_n, b_n are independent of x, is called a **trigonometric series.**

If we assume the possibility of expanding any arbitrary function $f(x)$ in a series of this form, and that it may be integrated term by term, we may readily find the coefficients a_n, b_n. Putting

$$(a) \qquad f(x) = \tfrac{1}{2}a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and integrating both sides between the limits $-\pi$ and $+\pi$, we find

$$\int_{-\pi}^{\pi} f(x)dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx = \pi a_0,$$

or

$$(b) \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,$$

since by the Calculus,

$$\int_{-\pi}^{\pi} \cos mx dx = 0, \quad \int_{-\pi}^{\pi} \sin mx dx = 0, \quad \text{if } m \neq 0.$$

Now multiply both sides of the equation (a) by $\cos mx$ and by $\sin mx$, and integrate term by term, and use the results (from the Calculus):

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0 \ (m \neq n), \\ \pi \ (m = n \neq 0), \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0 \ (m \neq n), \\ \pi \ (m = n \neq 0), \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx dx = 0.$$

We find

$$(c) \ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

A trigonometric series whose coefficients are given by relations (*b*), (*c*) is often called a **Fourier series.**

The preceding calculation is merely formal, on account of the assumptions made.

288. Fourier's Theorem.—Let us take the sum of the first $m + 1$ terms of the Fourier series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

then using the values of a_n and b_n from equations (*b*) and (*c*) of Art. **287**, we have, after simplifying,

$$S_{m+1} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha)[\tfrac{1}{2} + \cos (\alpha - x) + \cos 2(\alpha - x) + \ . \ . \ . \ + \cos m(\alpha - x)]d\alpha.$$

By a trigonometric formula,

$$\tfrac{1}{2} + \cos \theta + \cos 2\theta + \ . \ . \ . \ + \cos m\theta = \frac{\sin \dfrac{2m+1}{2}\theta}{2 \sin \dfrac{\theta}{2}},$$

so that

$$S_{m+1} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha) \cdot \frac{\sin \dfrac{2m+1}{2}(\alpha - x)}{2 \sin \dfrac{\alpha - x}{2}} d\alpha.$$

To determine the convergence of the Fourier series, we must find the limit of S_{m+1} as $m \rightarrow \infty$. To do this requires a somewhat lengthy and difficult discussion, into which we shall not enter; certain conditions, called **Dirichlet's conditions,** are imposed upon $f(x)$, and the final result of the discussion, for which we must refer to other works,[1] may be stated in the following proposition, called **Fourier's theorem:**

Theorem.—*Every function which is defined in the interval* $(-\pi, \pi)$ *and which satisfies Dirichlet's conditions in that interval may be represented by a Fourier's series in that interval.*

The further discussion of Fourier series is beyond the scope of this work.

REFERENCES

Goursat, E., "Course of Mathematical Analysis," arts. **195–198.**
Whittaker and Watson, "Modern Analysis," chap. IX.
Carslaw, "Fourier's Series and Fourier's Integrals," chap. V.
Hobson, "Theory of Functions of a Real Variable," chap. VII.
Pierpont, "Theory of Functions of Real Variables," vol. II, chap. XIII.
Byerly, W. E., "Fourier's Series and Spherical, Cylindrical and Ellipsoidal Harmonics."
Lebesgue, H., "Leçons sur les Séries Trigonométriques."
Knopp, "Unendliche Reihen," chap. XI.

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[1] See for example, Goursat's "Mathematical Analysis," vol. I, arts. **195–197.**

CHAPTER XXII

ASYMPTOTIC SERIES AND SUMMABLE SERIES

ASYMPTOTIC SERIES

289. Asymptotic Series.—*A divergent series of the form*

$$A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \ . \ . \ . \ + \frac{A_n}{z^n} + \ . \ . \ .$$

is said to be an **asymptotic representation** *of a function* $f(z)$ *for a given range of values of amplitude* z, *if we have*

$$\underset{|z| \to \infty}{L} z^n[f(z) - S_n(z)] = 0 \ (n \text{ fixed}),$$

where $S_n(z)$ *is the sum of the first* $(n + 1)$ *terms of the series.* We can express this:

$$\left|z^n[f(z) - S_n(z)]\right| < \epsilon$$

for $|z|$ sufficiently large, however small $\epsilon > 0$.

We indicate that the series is the asymptotic expansion of $f(z)$ by the notation

$$f(z) \sim \sum_{n=0}^{\infty} A_n z^{-n}.$$

290. Example of Asymptotic Expansion.—If we integrate $f(x) = \int_x^\infty t^{-1}e^{x-t}dt$ by parts repeatedly, we find

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \ . \ . \ . \ + (-1)^{n-1}\frac{(n-1)!}{x^n} + (-1)^n . n! \int_x^\infty \frac{e^{x-t}dt}{t^{n+1}}.$$

If we put

$$S_n(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \ . \ . \ . \ + (-1)^n \frac{n!}{x^{n+1}},$$

the series obtained by making $n \to \infty$ is divergent, as shown by the ratio test. Let n be fixed for the moment. We have

$$f(x) - S_n(x) = (-1)^{n+1}(n+1)! \int_x^\infty \frac{e^{x-t}dt}{t^{n+2}},$$

and since in this integral $t > x$, $e^{x-t} \leqq 1$, we have

$$|f(x) - S_n(x)| = (n+1)! \int_x^\infty \frac{e^{x-t}dt}{t^{n+2}} < (n+1)! \int_x^\infty \frac{dt}{t^{n+2}}$$
$$= \frac{n!}{x^{n+1}}.$$

Now for n fixed, we can find x so large that

$$x^n|f(x) - S_n(x)| < \epsilon$$

however small ϵ, so that the series is an asymptotic representation of $f(x)$. Hence, the value of the function $f(x)$ can be calculated with great accuracy for large values of x by taking the sum of a suitable number of terms of the series.

291. Uniqueness of an Asymptotic Expansion.—If $\sum_{n=0}^{\infty} a_n x^{-n}$ is the asymptotic representation of $f(x)$, we must have, for every n,

$$\underset{x}{L} x^n \left[f(x) - \sum_{i=0}^{n} a_i x^{-i} \right] = 0,$$

hence the coefficients a_i may be determined by the equations

$$\underset{x}{L} f(x) = a_0,$$
$$\underset{x}{L} x[f(x) - a_0] = a_1,$$
$$\underset{x}{L} x^2 \left[f(x) - a_0 - \frac{a_1}{x} \right] = a_2,$$

. . . .

If we apply these equations to the function $f(x) = e^{-x}$, we find

$$a_0 = \underset{x}{L} e^{-x} = 0,\ a_1 = \underset{x}{L} xe^{-x} = 0,\ a_2 = \underset{x}{L} x^2e^{-x} = 0, \text{ etc.},$$

so that the asymptotic expansion of e^{-x} consists of a series all of whose coefficients are zero.

It follows that if $f(x)$ possesses an asymptotic expansion, all functions $f(x) + Ce^{-x}$, where C is an arbitrary constant, also have the same asymptotic representation.

That is, a given series may represent asymptotically more than one function. But a given function cannot be represented by more than one distinct asymptotic expansion over the whole of a given range.

292. Multiplication of Asymptotic Series.—Let

$$f(x) \sim \sum_0^\infty A_m x^{-m},\ \phi(x) \sim \sum_0^\infty B_m x^{-m},$$

and let $S_n(x)$, $T_n(x)$ be the sums of their first $(n + 1)$ terms. Then

$$\underset{x}{L} x^n[f(x) - S_n(x)] = 0,\ \underset{x}{L} x^n[\phi(x) - T_n(x)] = 0,$$

or

$f(x) - S_n(x) = o(x^{-n}),\ \phi(x) - T_n(x) = o(x^{-n}).$
Put $C_m = A_0B_m + A_1B_{m-1} + \ . \ . \ . \ + A_mB_0$, then

$$S_n(x) \cdot T_n(x) = \sum_{m=0}^{n} C_m x^{-m} + o(x^{-n}).$$

Now

$$\begin{aligned} f(x) \cdot \phi(x) &= [S_n(x) + o(x^{-n})] \cdot [T_n(x) + o(x^{-n})] \\ &= S_n(x) \cdot T_n(x) + o(x^{-n}) \\ &= \sum_0^n C_m x^{-m} + o(x^{-n}). \end{aligned}$$

Hence,

$$f(x) \cdot \phi(x) \sim \sum_0^\infty C_m x^{-m}.$$

293. Integration of Asymptotic Series.—*An asymptotic series may be integrated term by term, giving an asymptotic series which is the representation of the integral of the function represented by the original series.*

For, if

$$f(x) \sim \sum_{m=2}^\infty A_m x^{-m},$$

and

$$S_n(x) = \sum_{m=2}^n A_m x^{-m},$$

then for any arbitrarily small $\epsilon > 0$, we can find x_0 so that

$$|f(x) - S_n(x)| < \epsilon |x|^{-n},\ x > x_0,$$

hence

$$\left| \int_x^\infty f(x)dx - \int_x^\infty S_n(x)dx \right| \leqq \int_x^\infty |f(x) - S_n(x)|dx$$

$$< \frac{\epsilon}{(n-1)x^{n-1}}.$$

Since

$$\int_x^\infty S_n(x)dx = \frac{A_2}{x} + \frac{A_3}{2x^2} + \cdots + \frac{A_n}{(n-1)x^{n-1}},$$

we have

$$\int_x^\infty f(x)dx \sim \sum_{m=2}^\infty \frac{A_m}{(m-1)x^{m-1}}.$$

SUMMABLE SERIES

294. Simple Indeterminacy.—Let $\sum a_n$ be any given series, and put

$$S_n = a_0 + a_1 + a_2 + \ . \ . \ . + a_n = \sum_0^n a_i,$$

$$\sigma_n = \frac{S_0 + S_1 + \ . \ . \ . + S_n}{n+1}.$$

It may happen that $\underset{n}{L}\sigma_n$ may exist even when $\underset{n}{L}S_n$ does not, *i.e.*, when $\sum a_n$ is not convergent. *When $L\sigma_n$ exists with a finite value S, we say the series $\sum a_n$* **is simply indeterminate** *and that S is its* **Sum.**

Example.—For the oscillating series $1 - 1 + 1 - 1 + 1 - 1 + \ . \ . \ .$, we have

$$\sigma_n = \frac{1 + 0 + 1 + 0 + \ . \ . \ . \text{ to } (n+1) \text{ terms}}{n+1}$$

$$= \frac{1}{2} \text{ or } \frac{\frac{1}{2} n + 1}{n+1},$$

according as n is odd or even, so that $L\sigma_n = \frac{1}{2}$, and the series is simply indeterminate with Sum $\frac{1}{2}$.

295. Theorem.—*If $\sum a_n$ is convergent with sum S, it is also simply indeterminate with Sum S.*

For if $S = LS_n$, then by Art. **39** we have

$$L \frac{1}{n+1} (S_0 + S_1 + \ . \ . \ . + S_n) = S,$$

so that $\sum a_n$ is simply indeterminate with Sum S.

296. Theorem.—*A properly divergent series cannot be simply indeterminate with finite Sum.*

If $S_n \rightarrow +\infty$, then

$$S_n > G, n \geqq p,$$

where G is arbitrarily large. Writing

$$\sigma_n = \frac{S_0 + S_1 + \ .\ .\ .\ + S_p}{n+1} + \frac{S_{p+1} + S_{p+2} + \ .\ .\ .\ + S_n}{n+1},$$

we have

$$\frac{S_0 + S_1 + \ .\ .\ .\ + S_p}{n+1} > -\frac{p}{n+1}\cdot G,$$
$$\frac{S_{p+1} + \ .\ .\ .\ + S_n}{n+1} > \frac{n-p}{n+1}\cdot G,$$
$$\therefore \sigma_n > -\frac{p}{n+1}G + \frac{n-p}{n+1}G = \frac{n-2p}{n+1}\cdot G.$$

If $n \geqq 4p + 1$,

$$\sigma_n \geqq \tfrac{1}{2}G,$$

hence $\sigma_n \rightarrow +\infty$ as $n \rightarrow \infty$, and our theorem follows.

297. Cesàro's Theorem on the Multiplication of Convergent Series.—*The product of two convergent series, formed according to Cauchy's rule, is a series which is either convergent or simply indeterminate, and the sum or Sum is equal to the product of the sums of the factor series.*

Let $\sum\limits_0^\infty u_n$, $\sum\limits_0^\infty v_n$ be the given convergent series, and put

$$w_n = u_0v_n + u_1v_{n-1} + \ .\ .\ .\ + u_nv_0,$$

and

$$U_n = \sum_0^n u_i,\ V_n = \sum_0^n v_i,\ W_n = \sum_0^n w_i.$$

Then we find

$$W_n = u_0V_n + u_1V_{n-1} + \ .\ .\ .\ + u_nV_0,$$

$$(a)\quad W_0 + W_1 + \ .\ .\ .\ + W_n = U_0V_n + U_1V_{n-1} + \ .\ .\ .\ + U_nV_0.$$

Now by Art. **40**,

$$L\frac{1}{n+1}(U_0V_n + U_1V_{n-1} + \ . \ . \ . \ + U_nV_0) = UV,$$

since $U_n \to U$, $V_n \to V$. Hence, by (*a*),

$$L\frac{1}{n+1}(W_0 + W_1 + \ . \ . \ . \ + W_n) = UV,$$

and $\sum\limits_0^\infty w_n$ is simply indeterminate with Sum UV.

298. Theorem.—*A necessary condition that the series* $\sum a_n$ *be simply indeterminate is* $L\dfrac{a_n}{n} = 0$.

For if

$$L\frac{S_0 + S_1 + \ . \ . \ . \ + S_n}{n+1} = S,$$

we may write

$$S_0 + S_1 + \ . \ . \ . \ + S_n = (n+1)S + (n+1)\delta_n,$$

where $\delta_n \to 0$ as $n \to \infty$. Then

$$S_0 + S_1 + \ . \ . \ . \ + S_{n-1} = nS + n\delta_{n-1}.$$

Subtracting,

$$S_n = S + (n+1)\delta_n - n\delta_{n-1},$$

$$\therefore a_n = S_n - S_{n-1} = (n+1)\delta_n - 2n\delta_{n-1} + (n-1)\,\delta_{n-2},$$

or

$$\frac{a_n}{n} = \left(1 + \frac{1}{n}\right)\delta_n - 2\delta_{n-1} + \left(1 - \frac{1}{n}\right)\delta_{n-2}.$$

Since $\delta_n \to 0$ as $n \to \infty$, we get

$$L\frac{a_n}{n} = 0.$$

299. Cesàro Summability.—Let $\sum\limits_0^\infty a_n$ be any given series.

Put

$$S_n = \sum_{i=0}^{n} a_i,$$

$$S_n^{(k)} = \sum_{i=0}^{n} \binom{k+n-i-1}{n-i} S_i,$$

$$A_n^{(k)} = \binom{k+n}{n} = \frac{(k+1)(k+2)\ .\ .\ .\ (k+n)}{n!},$$

$$C_n^{(k)} = \frac{S_n^{(k)}}{A_n^{(k)}}.$$

If $\underset{n}{L} C_n^{(k)}$ *exists with a finite value* S, *we say that the series* $\sum a_n$ *is* **summable** (**C, k**) *and that* S *is its* **Sum.**

Summability $(C, 0)$ is convergency; summability $(C, 1)$ is simple indeterminacy.[1]

300. Theorem.—*If* $\sum a_n$ *is summable* (C, k), *it is also summable* $(C, k+1)$ *with the same Sum.*

If we multiply the series $\sum_{0}^{\infty} S_n x^n$ by the series for $(1-x)^{-k}$, the coefficient of x^n in the product series will be found to be $S_n^{(k)}$, so that

$$\sum_{0}^{\infty} S_n^{(k)} x^n = (1-x)^{-k} \sum_{0}^{\infty} S_n x^n.$$

Similarly we find

$$\sum_{0}^{\infty} S_n x^n = (1-x)^{-1} \cdot \sum_{0}^{\infty} a_n x^n,$$

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[1] The theory of this method of summation can be developed from this definition for *any* value of k, but we shall consider only positive integral values.

hence

$$\sum_0^\infty S_n^{(k)} x^n = (1-x)^{-(k+1)} \sum_0^\infty a_n x^n.$$

Also

$$\sum_0^\infty A_n^{(k)} x^n = (1-x)^{-(k+1)}.$$

These series all converge absolutely for $|x| < 1$, if $\sum a_n$ is summable (C, k). For, by the last equation, $\sum_0^\infty A_n^{(k)} x^n$ converges absolutely for $|x| < 1$; and if $LC_n^{(k)}$ exists, we have

$$|C_n^{(k)}| < \text{some constant } K,$$

or

$$|S_n^{(k)}| < K \cdot A_n^{(k)},$$

so that $\sum_0^\infty S_n^{(k)} x^n$ converges absolutely when $|x| < 1$.

Expanding the identity

$$(1-x)^{-(k+1)} = (1-x)^{-1} . (1-x)^{-k},$$

and comparing coefficients of x^n, we obtain

$$A_n^{(k)} = \sum_0^n A_i^{(k-1)}.$$

Also

$$\sum_0^\infty S_n^{(k+1)} x^n = (1-x)^{-(k+2)} \sum_0^\infty a_n x^n$$

$$= (1-x)^{-1}(1-x)^{-(k+1)} \sum_0^\infty a_n x^n = \sum_0^\infty x^n \cdot \sum_0^\infty S_n^{(k)} x^n;$$

from this we find

$$S_n^{(k+1)} = \sum_0^n S_i^{(k)}.$$

Then we have

$$S_n^{(k)} = S_n^{(k+1)} - S_{n-1}^{(k+1)},$$
$$A_n^{(k)} = A_n^{(k+1)} - A_{n-1}^{(k+1)}.$$

Applying the Cauchy-Stolz theorem (Art. **38**), we get

$$\underset{n}{L} \frac{S_n^{(k+1)}}{A_n^{(k+1)}} = \underset{n}{L} \frac{S_n^{(k)}}{A_n^{(k)}},$$

which proves our theorem.

Since convergency is summability $(C, 0)$, we see from this result that:

If $\sum a_n$ *is convergent with sum* S, *it is also summable* (C, k) *with the Sum* S, *for any integral value of* k.[1]

301. Theorem.—*If* $\sum_0^\infty a_n$ *is summable* (C, k) *with Sum* S, *then*

$$\underset{x \to 1}{L} \sum_0^\infty a_n x^n = S.$$

By a limit theorem due to Cesàro:[2]

$$\underset{x \to 1}{L} \frac{\sum_0^\infty b_n x^n}{\sum_0^\infty c_n x^n} = \underset{n}{L} \frac{\sum_0^n b_i,}{\sum_0^n c_i},$$

provided the latter limit exists, $\sum c_n$ being a series of positive terms.

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[1] It can be easily shown that a properly divergent series cannot be summable (C, k) with finite sum.

[2] See Bromwich, "Infinite Series," art. **51**.

Now

$$\frac{\sum\limits_0^\infty S_n^{(k)}x^n}{\sum\limits_0^\infty A_n^{(k)}x^n} = \frac{(1-x)^{-(k+1)}\sum\limits_0^\infty a_n}{(1-x)^{-(k+1)}} = \sum_0^\infty a_n x^n.$$

Hence, by the above theorem, we have

$$\underset{x\to 1}{L}\sum_0^\infty a_n x^n = \underset{x\to 1}{L}\frac{\sum\limits_0^\infty S_n^{(k)}x^n}{\sum\limits_0^\infty A_n^{(k)}x^n} = \underset{n}{L}\frac{\sum\limits_0^n S_i^{(k)}}{\sum\limits_0^n A_i^{(k)}} = \underset{n}{L}\frac{S_n^{(k+1)}}{A_n^{(k+1)}} = S,$$

provided $\sum a_n$ is summable (C, k).

This result is a generalization of Abel's theorem (Art. **171**). The special case where $k = 1$ (simple indeterminacy) is called Frobenius' theorem.

302. Cesàro's Multiplication Theorem.—*If* $\sum u_n$ *is summable* (C, r) *with Sum* U, *and* $\sum v_n$ *is summable* (C, s) *with Sum* V, *then the Cauchy product series* $\sum w_n$ *is summable* $(C, r + s + 1)$ *with Sum* UV.

The proof of this theorem is too long to give here.[1]

303. Borel Summability.—*The series* $\sum a_n$ *is said to be* **summable** (B) *with* **Sum** S *(by Borel's method) if the integral*

$$\int_0^\infty e^{-x}U(x)dx$$

converges with the value S, *where* $U(x) = \sum\limits_0^\infty a_n \cdot \dfrac{x^n}{n!}$.

Examples.—(1) For the series $1 - 1 + 1 - 1 + \ldots$, we have

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[1] See Bromwich, "Infinite Series," art. **106**.

$$U(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ . \ . \ . \ = e^{-x},$$

$$\therefore \int_0^\infty e^{-x}U(x)dx = \int_0^\infty e^{-2x}dx = \tfrac{1}{2}.$$

(2) For the series $1 - t + t^2 - t^3 + \ . \ . \ . \ (t > 1)$, we have

$$U(x) = 1 - tx + \frac{t^2x^2}{2!} - \frac{t^3x^3}{3!} + \ . \ . \ . \ = e^{-tx},$$

$$\therefore \int_0^\infty e^{-x}U(x)dx = \int_0^\infty e^{-(t+1)x}dx = \frac{1}{1+t}.$$

(3) For the series $1 - 2 + 3 - 4 + 5 - \ . \ . \ . \ ,$

$$U(x) = 1 - 2x + 3 \cdot \frac{x^2}{2!} - 4 \cdot \frac{x^3}{3!} + \ . \ . \ . \ = e^{-x}(1 - x),$$

$$\therefore \int_0^\infty e^{-x}U(x)dx = \int_0^\infty e^{-2x}(1 - x)dx = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}.$$

304. Theorem.—*A convergent series is summable (B) with Sum equal to the sum.*

For if $\sum a_n$ is convergent with sum S,

$$|a_n| < \epsilon, \ n \geqq m.$$

Now

$$\left|\sum_0^\infty a_n \cdot \frac{x^n}{n!}\right| \leqq \sum_0^{m-1} |a_n| \cdot \frac{x^n}{n!} + \sum_m^\infty |a_n| \frac{x^n}{n!}$$

$$< \sum_0^{m-1} |a_n| \cdot \frac{x^n}{n!} + \epsilon \cdot \sum_m^\infty \frac{x^n}{n!}$$

$$\therefore |e^{-x}U(x)| < e^{-x} \sum_0^{m-1} |a_n| \cdot \frac{x^n}{n!} + \epsilon \cdot e^{-x}\left(\sum_0^\infty \frac{x^n}{n!} - \sum_0^{m-1} \frac{x^n}{n!}\right),$$

$$< e^{-x} \sum_0^{m-1} |a_n| \cdot \frac{x^n}{n!} + \epsilon - \epsilon \cdot e^{-x} \sum_0^{m-1} \frac{x^n}{n!}$$

By Art. **59**, if m be kept fixed,

$$\underset{x}{L}e^{-x}\sum_{0}^{m-1}|a_n|\cdot\frac{x^n}{n!} = 0,$$

$$\underset{x}{L}e^{-x}\sum_{0}^{m-1}\frac{x^n}{n!} = 0$$

hence

$$\underset{x}{L}e^{-x}U(x) = 0.$$

Since $\sum a_n$ is convergent with sum S, we may write

$$S_n = S - R_n,\quad |R_n| < \epsilon,\quad n \geqq m.$$

Then

$$e^{-x}S(x) \equiv e^{-x}\sum_{0}^{\infty}S_n\frac{x^n}{n!} = S - e^{-x}\sum_{0}^{\infty}R_n\cdot\frac{x^n}{n!}$$

$$e^{-x}\left|\sum_{0}^{\infty}R_n\frac{x^n}{n!}\right| \leqq e^{-x}\sum_{0}^{m-1}\left|R_n\right|\cdot\frac{x^n}{n!} + e^{-x}\sum_{m}^{\infty}\left|R_n\right|\frac{x^n}{n!},$$

$$< e^{-x}\sum_{0}^{m-1}\left|R_n\right|\cdot\frac{x^n}{n!} + \epsilon e^{-x}\sum_{m}^{\infty}\frac{x^n}{n!}$$

$$< e^{-x}\sum_{0}^{m-1}\left|R_n\right|\frac{x^n}{n!} + \epsilon.$$

But by Art. **59**, we have

$$\underset{x}{L}e^{-x}\sum_{0}^{m-1}\left|R_n\right|\cdot\frac{x^n}{n!} = 0,$$

hence

$$\underset{x}{L}e^{-x}S(x) = S.$$

Now put $S(x) = \sum_{0}^{\infty}S_n\frac{x^n}{n!}$, $S_n = \sum_{0}^{n}a_i$, then

$$\int S(x)dx = \sum_{0}^{\infty}S_n\cdot\frac{x^{n+1}}{(n+1)!} = \sum_{1}^{\infty}S_{n-1}\cdot\frac{x^n}{n!},$$

$$\therefore S(x) - \int S(x)dx = S_0 + \sum_1^\infty (S_n - S_{n-1}) \frac{x^n}{n!}$$

$$= \sum_0^\infty a_n \cdot \frac{x^n}{n!} = U(x).$$

Differentiating,

$$S'(x) - S(x) = U'(x),$$

$$\therefore e^{-x}[S'(x) - S(x)] = e^{-x}[U'(x) - U(x)] + e^{-x}U(x).$$

Now $e^{-x}[S'(x) - S(x)]$ is the derivative of $e^{-x}S(x)$ and $e^{-x}[U'(x) - U(x)]$ is the derivative of $e^{-x}U(x)$, hence integrating, we have

$$e^{-x}S(x) = e^{-x}U(x) + \int_0^x e^{-x}U(x)dx.$$

But we have already shown that

$$\underset{x}{L}e^{-x}S(x) = S,\ \underset{x}{L}e^{-x}U(x) = 0,$$

hence

$$\int_0^\infty e^{-x}U(x)dx = S,$$

and our theorem is proved.

It can be shown that *a properly divergent series cannot be summable* (*B*) *with finite Sum.*

305. Absolute Summability (**B**).—*The series* $\sum a_n$ *is called* **absolutely summable** (**B**) *if the integrals*

$$\int_0^\infty e^{-x}U(x)dx \text{ and } \int_0^\infty e^{-x}U^{(k)}(x)dx$$

are all absolutely convergent, where k denotes any index of differentiation.

It has been proved by Borel that *if two series* $\sum u_n$ *and* $\sum v_n$ *are absolutely summable* (*B*), *then the product series* $\sum w_n$ *is also absolutely summable* (*B*), *and the relation between their Sums is* $UV = W$.

306. Uniform Summability.—*If the terms of the given series are functions of a variable* α, *we call the series* **uniformly summable** (**B**) *with respect to* α *if the integral*

$$\int_0^\infty e^{-x}U(x, \alpha)dx$$

converges uniformly with respect to α.

By means of this concept of uniform summability we may discuss continuity, differentiation and integration in a manner similar to that for uniform convergence.

Frobenius' theorem may be extended to this method of summation.

The applications of Borel's method to the study of power series are of the greatest importance. It is found that power series are summable (B) in a so-called **polygon of summability**, which in general extends beyond the circle of convergence.

307. The Problem of Non-convergent Series.—It will be seen from the discussions of this chapter that the definitions of asymptotic series and summability give methods for assigning a meaning to non-convergent series. The problem is to find some methods of associating by a definite law with each non-convergent series, of as extended a class as possible, a definite number, called the Sum of the series, which can be used as a substitute for the series in the usual calculations with series, and which will obey the same laws as do convergent series, at least as far as possible. In this chapter we have discussed very briefly only a few of the many definitions of summability which have been proposed.

REFERENCES

Borel, E., "Leçons sur les Séries Divérgentes."

Bromwich, "Theory of Infinite Series," chap. XI.

Whittaker and Watson, "Modern Analysis," chap. VIII.

Vivanti-Gutzmer, "Theorie der eindeutigen analytischen Funktionen," pp. 319–333.

Knopp, "Unendliche Reihen," chap. XIII.

CHAPTER XXIII

INFINITE CONTINUED FRACTIONS

308. Finite Continued Fraction.—An expression of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots \cfrac{a_{n-1}}{b_{n-1} + \cfrac{a_n}{b_n}}}}}$$

is called a **finite continued fraction.** It may be denoted more compactly by

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n};$$

other forms of notation sometimes used are

$$b_0 \dot{+} \frac{a_1}{b_1} \dot{+} \frac{a_2}{b_2} \dot{+} \cdots \dot{+} \frac{a_n}{b_n},$$

or

$$b_0 + \frac{a_1}{b_1 +}\,\frac{a_2}{b_2 +} \cdots \frac{a_n}{b_n},$$

or

$$\left(b_0; \frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right),$$

or

$$\left[b_0; + \frac{a_i}{b_i}\right]_1^n, \qquad \text{or } \left[b_0 + \frac{a_i}{b_i}\right]_1^n.$$

The a_i, b_i are called the **elements** of the continued fraction, b_0 is the first term, and a_i, b_i are the i^{th} **partial numerator** and **partial denominator.**

If in the continued fraction of the n^{th} order,

$$K_n = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \quad . \quad . \quad . \quad + \frac{a_n|}{|b_n},$$

we take successively $n = 0, 1, 2, \ . \ . \ . \ ,$ we find

$$K_0 = b_0 = \frac{b_0}{1},$$

$$K_1 = b_0 + \frac{a_1}{b_1} = \frac{b_0 b_1 + a_1}{b_1},$$

$$K_2 = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2}} = \frac{b_0b_1b_2 + a_1b_2 + b_0a_2}{b_1b_2 + a_2},$$

etc.,

so that it is evident that K_p can be expressed as an ordinary fraction $\frac{A_p}{B_p}$, where A_p and B_p are functions of the a_i and b_i (up to $i = p$). These fractions $\frac{A_p}{B_p}$ are called the **convergents** of the continued fraction K_n, and the n^{th} convergent $\frac{A_n}{B_n}$ is equal to the continued fraction K_n itself.

309. Infinite Continued Fraction.—Let (a_n) and (b_n) be any two infinite sequences. Form the continued fraction of the n^{th} order

$$K_n = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \quad . \quad . \quad . \quad + \frac{a_n|}{|b_n},$$

whose n^{th} convergent is $\frac{A_n}{B_n}\cdot$ If we now let $n \to \infty$, we obtain an **infinite continued fraction**

$$K = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \ . \ . \ . \ + \frac{a_n|}{|b_n} + \ . \ . \ . \ (\text{to } \infty),$$

which may also be written

$$K = \left[b_0 \overset{,}{+} \frac{a_i}{b_i}\right]_1^\infty;$$

and we have an infinite sequence of convergents

$$\frac{A_0}{B_0}, \frac{A_1}{B_1}, \frac{A_2}{B_2}, \ . \ . \ . \ , \frac{A_n}{B_n}, \ . \ . \ .$$

The continued fraction K is called **convergent, divergent** (**properly** or **improperly**), or **oscillating,** *according as the sequence of convergents*

$$\frac{A_0}{B_0}, \frac{A_1}{B_1}, \frac{A_2}{B_2}, \ . \ . \ . \ , \frac{A_n}{B_n}, \ . \ . \ .$$

is convergent, divergent (properly or improperly), or oscillating.

Thus, the continued fraction is convergent if

$$\underset{n}{L}\, \frac{A_n}{B_n} = K$$

exists with a finite value K, and K is called the *value* of the continued fraction.

The continued fraction is divergent if

$$L \frac{A_n}{B_n} = \infty;$$

it is properly divergent if this limit is $+\infty$ or $-\infty$, otherwise improperly divergent.

The continued fraction is oscillating if $\frac{A_n}{B_n}$ has no limit, finite or infinite.

310. Recursion Formulae.—Since

$$\frac{A_i}{B_i} = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \quad . \quad . \quad . \quad + \frac{a_i|}{|b_i},$$

we have

$$\begin{aligned} &\frac{A_0}{B_0} = b_0 = \frac{b_0}{1}, \\ \therefore\ &A_0 = b_0,\ B_0 = 1; \\ &\frac{A_1}{B_1} = b_0 + \frac{a_1}{b_1} = \frac{b_0b_1 + a_1}{b_1}, \\ \therefore\ &A_1 = b_0b_1 + a_1,\ B_1 = b_1; \end{aligned}$$

we obtain $\frac{A_2}{B_2}$ from $\frac{A_1}{B_1}$ by replacing b_1 by $b_1 + \frac{a_2}{b_2}$, hence

$$\frac{A_2}{B_2} = \frac{b_0\left(b_1 + \frac{a_2}{b_2}\right) + a_1}{b_1 + \frac{a_2}{b_2}} = \frac{(b_0b_1 + a_1)a_2 + b_0a_2}{b_1b_2 + a_2},$$

$$\therefore A_2 = (b_0b_1 + a_1)a_2 + b_0a_2,\ B_2 = b_1b_2 + a_2,$$

or

$$A_2 = b_2A_1 + a_2A_0,\ B_2 = b_2B_1 + a_2B_0.$$

Generally we have the recursion formulae

$$(a) \qquad \begin{cases} A_n = b_nA_{n-1} + a_nA_{n-2}, \\ B_n = b_nB_{n-1} + a_nB_{n-2}. \end{cases}$$

These may be easily proved by induction.

An important property of any two consecutive convergents is

$$(b) \qquad A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}a_1a_2 \quad . \quad . \quad . \quad a_n.$$

For, by (a),

$$\begin{aligned} A_{n+1}B_n - A_nB_{n+1} &= (b_{n+1}A_n + a_{n+1}A_{n-1})B_n \\ &\qquad - A_n(b_{n+1}B_n + a_{n+1}B_{n-1}) \\ &= -(A_nB_{n-1} - A_{n-1}B_n)a_{n+1}. \end{aligned}$$

Then if we assume (b) to hold for n, we have

$$A_{n+1}B_n - A_nB_{n+1} = (-1)^n \cdot a_1a_2 \ . \ . \ . \ a_n \cdot a_{n+1},$$

which is the same form as (b) except that it has everywhere $n + 1$ in place of n in the former; we have also

$$A_1B_0 - A_0B_1 = (b_0b_1 + a_1) - b_0b_1 = a_1 = (-1)^0a_1;$$

it follows by induction that formula (b) holds for every n.

From (b) we get

$$(c) \qquad \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = (-1)^{n-1} \cdot \frac{a_1a_2 \ . \ . \ . \ a_n}{B_{n-1}\ B_n}.$$

We have also

$$\begin{aligned} A_nB_{n-2} - A_{n-2}B_n &= (b_nA_{n-1} + a_nA_{n-2})B_{n-2} \\ &\quad - A_{n-2}(b_nB_{n-1} + a_nB_{n-2}) \\ &= b_n(A_{n-1}B_{n-2} - A_{n-2}B_{n-1}), \end{aligned}$$

$$\therefore A_nB_{n-2} - A_{n-2}B_n = (-1)^{n-2}b_n \cdot a_1a_2 \ . \ . \ . \ a_n, \text{ by } (b).$$

$$(d) \qquad \therefore \frac{A_n}{B_n} - \frac{A_{n-2}}{B_{n-2}} = (-1)^n \frac{b_na_1a_2 \ . \ . \ . \ a_n}{B_{n-2}\ B_n}.$$

311. Equivalent Series.—*A series* $\sum c_n$ *and a continued fraction* $b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \ . \ . \ . \ + \frac{a_n|}{|b_n} + \ . \ . \ .$ *are said to be* **equivalent** *when*

$$c_0 + c_1 + c_2 + \ . \ . \ . \ + c_n = S_n = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \ . \ . \ . \ + \frac{a_n|}{|b_n} = \frac{A_n}{B_n}$$

for every n.

Using (c) of Art. **310,** we find

$$\frac{A_n}{B_n} = \frac{A_0}{B_0} + \left(\frac{A_1}{B_1} - \frac{A_0}{B_0}\right) + \left(\frac{A_2}{B_2} - \frac{A_1}{B_1}\right) + \ . \ . \ . \ + \left(\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}}\right)$$

$$= b_0 + \frac{a_1}{B_0B_1} - \frac{a_1a_2}{B_1B_2} + \frac{a_1a_2a_3}{B_2B_3} - \cdots + (-1)^{n-1}\frac{a_1a_2 \cdots a_n}{B_{n-1}B_n}.$$

The infinite series

$$b_0 + \frac{a_1}{B_0B_1} - \frac{a_1a_2}{B_1B_2} + \cdots + (-1)^{n-1}\frac{a_1a_2 \cdots a_n}{B_{n-1}B_n} + \cdots$$

is then the equivalent series of the infinite continued fraction

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n} + \cdots$$

Both series and continued fraction converge and diverge simultaneously.

A given infinite series $\sum\limits_0^\infty c_n$ may be converted into an equivalent continued fraction as follows: If S_n is the sum of the first $n + 1$ terms of $\sum c_i$: $S_n = \sum\limits_0^n c_i$, then for the series to be equivalent to the continued fraction

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n} + \cdots$$

we must have, for every n,

$$\frac{S_n}{1} = \frac{A_n}{B_n}.$$

Using the formulae (a) of Art. **310,** we get

$$\begin{aligned} &S_0 = b_0,\ S_1 = b_0b_1 + a_1,\ 1 = b_1, \\ &\left.\begin{aligned} S_n &= b_nS_{n-1} + a_nS_{n-2} \\ 1 &= b_n \cdot 1 + a_n \cdot 1 \end{aligned}\right\} (n = 2, 3, 4, \ldots). \end{aligned}$$

Solving for b_n, a_n, we find

$$b_0 = S_0,\ a_1 = S_1 - S_0,\ b_1 = 1,$$

$$\left.\begin{aligned} a_n &= \frac{S_{n-1} - S_n}{S_{n-1} - S_{n-2}} \\ b_n &= \frac{S_n - S_{n-2}}{S_{n-1} - S_{n-2}} \end{aligned}\right\} (n = 2, 3, 4, \ . \ . \ . \ ).$$

But $S_0 = c_0$, $S_1 - S_0 = c_1$, and $S_{n-1} - S_n = -c_n$, $S_{n-1} - S_{n-2} = c_{n-1}$, $S_n - S_{n-2} = c_{n+1} + c_n$, hence

$$b_0 = c_0,\ b_1 = 1,\ a_1 = c_1,$$

$$a_n = -\frac{c_n}{c_{n-1}},$$

$$b_n = 1 + \frac{c_n}{c_{n-1}}.$$

We have then the following

Theorem.—*The infinite series* $c_0 + c_1 + c_2 + \ . \ . \ . \ +$ $c_n + \ . \ . \ .$ *is equivalent to the infinite continued fraction*

$$c_0 + \frac{c_1|}{|1} - \frac{\frac{c_2}{c_1}\bigg|}{\bigg|1 + \frac{c_2}{c_1}} - \frac{\frac{c_3}{c_2}\bigg|}{\bigg|1 + \frac{c_3}{c_2}} - \ . \ . \ . \ - \frac{\frac{c_n}{c_{n-1}}\bigg|}{\bigg|1 + \frac{c_n}{c_{n-1}}} - \ . \ . \ .$$

312. Equivalent Product.—*An infinite product* $\prod_0^{\infty}(1 + u_n)$ *and an infinite continued fraction*

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \ . \ . \ . \ + \frac{a_n|}{|b_n} + \ . \ . \ .$$

are said to be **equivalent** *when*

$$(1 + u_0)(1 + u_1)(1 + u_2) \ . \ . \ . \ (1 + u_n) = P_n$$
$$= b_0 + \frac{a_1|}{|b_1} + \ . \ . \ . \ + \frac{a_n|}{|b_n} = \frac{A_n}{B_n}$$

for every n.

To convert this product into its equivalent continued fraction, we may use the formulae of Art. **311,** putting P_n in place of S_n; we then have

$$\begin{aligned} b_0 &= P_0 = (1 + u_0), \\ a_1 &= P_1 - P_0 = (1 + u_0)u_1,\ b_1 = 1, \end{aligned}$$

$$\left.\begin{aligned} a_n &= \frac{P_{n-1} - P_n}{P_{n-1} - P_{n-2}} = -\left(1 + u_{n-1}\right) \cdot \frac{u_n}{u_{n-1}} \\ b_n &= \frac{P_n - P_{n-2}}{P_{n-1} - P_{n-2}} = 1 + \left(1 + u_{n-1}\right) \cdot \frac{u_n}{u_{n-1}} \end{aligned}\right\} (n \geqq 2).$$

Hence we have the

Theorem.—*The product* $\prod\limits_0^\infty (1 + u_n)$ *is equivalent to the continued fraction*

$$1 + u_0 + \frac{(1 + u_2)u_1|}{|\quad 1} - \frac{\alpha_2\ |}{|1 + \alpha_2} - \frac{\alpha_3\ |}{|1 + \alpha_3} - \frac{\alpha_4\ |}{|1 + \alpha_4} - \ \cdot\ \cdot\ \cdot\ ,$$

where

$$\alpha_n = \left(1 + u_{n-1}\right)\frac{u_n}{u_{n-1}}.$$

The product and continued fraction converge or diverge simultaneously.

313. Theorem.—*If the series* $\sum b_n$ *is absolutely convergent, the continued fraction*

$$b_0 + \frac{1|}{|b_1} + \frac{1|}{|b_2} + \ \cdot\ \cdot\ \cdot\ + \frac{1|}{|b_n} + \ \cdot\ \cdot\ \cdot$$

is improperly divergent or oscillating.

If m is any fixed value, then for $n > m$, we have

$$\frac{A_n}{B_n} = \frac{A_m}{B_m} + \left(\frac{A_{m+1}}{B_{m+1}} - \frac{A_m}{B_m}\right) + \left(\frac{A_{m+2}}{B_{m+2}} - \frac{A_{m+1}}{B_{m+1}}\right) + \ \cdot\ \cdot\ \cdot\ + \left(\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}}\right)$$

$$= \frac{A_m}{B_m} + (-1)^m \cdot \frac{1}{B_m B_{m+1}} + (-1)^{m+1} \frac{1}{B_{m+1} B_{m+2}} + \cdots + (-1)^{n-1} \cdot \frac{1}{B_{n-1} B_n} \text{ (by Art. } \mathbf{310}\text{).}$$

In order that the continued fraction be convergent, it is necessary that this equivalent series be convergent, and a necessary condition for the convergence of this series is that its n^{th} term approach 0 as $n \to \infty$, or that

$$(a) \qquad \underset{n}{L} |B_{n-1} B_n| = \infty.$$

Now it can be easily shown that

$$|B_n| \leqq \prod_{i=1}^{n} (1 + |b_i|) < \prod_{1}^{\infty} (1 + |b_i|),$$

since the infinite product converges because of the convergence of the series $\sum |b_i|$ (Art. **250**). But this result shows that (a) cannot be satisfied, hence the continued fraction cannot be convergent. That the continued fraction cannot be properly divergent, may be shown indirectly. If the continued fraction were properly divergent, we should have $L\frac{A_n}{B_n} = \infty$ or $L\frac{B_n}{A_n} = 0$, and the continued fraction $\frac{1}{b_0} + \frac{1|}{|b_1} + \cdots$ would converge to 0, which is not possible, since $\sum b_n$ is absolutely convergent.

314. Theorem.—*The continued fraction* $\left[b_0 + \frac{a_n}{b_n}\right]_1^\infty$ *diverges improperly when each of the series*

$$\sum_1^\infty \frac{a_1 a_3 \ldots a_{2n-1}}{a_2 a_4 \ldots a_{2n}} \cdot b_{2n}, \quad \sum_0^\infty \frac{a_2 a_4 \ldots a_{2n}}{a_1 a_3 \ldots a_{2n-1}} \cdot \frac{b_{2n+1}}{a_{2n+1}}$$

converges absolutely.

It can be shown without much difficulty, by use of Art. **310,** (*a*), that the two continued fractions

$$(a)\qquad b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n} + \cdots$$

and

$$(b)\quad b_0 + \frac{c_1a_1|}{|c_1b_1} + \frac{c_1c_2a_2|}{|c_2b_2} + \cdots + \frac{c_{n-1}c_na_n|}{|c_nb_n} + \cdots$$

where $c_1, c_2, \ldots, c_n, \ldots$ are arbitrary numbers $\neq 0$, are equivalent, that is, that their n^{th} convergents are equal, for every n.

Now choose the c's so that

$c_1a_1 = 1,\ c_1c_2a_2 = 1,\ c_2c_3a_3 = 1, \ldots, c_{n-1}c_na_n = 1, \ldots,$ then solving for the c's, we have

$$c_1 = \frac{1}{a_1},\ c_{2i} = \frac{a_1a_3 \ldots a_{2i-1}}{a_2a_4 \ldots a_{2i}},\ c_{2i+1} = \frac{a_2a_4 \ldots a_{2i}}{a_1a_3 \ldots a_{2i+1}}$$
$$(i = 1, 2, \ldots).$$

Then the continued fraction (*b*) becomes

$$(c)\qquad b_0 + \frac{1\ |}{|c_1b_1} + \frac{1\ |}{|c_2b_2} + \frac{1\ |}{|c_3b_3} + \cdots + \frac{1\ |}{|c_nb_n} + \cdots$$

Now apply the theorem of Art. **313** to this continued fraction, and our theorem follows.

315. Continued Fractions with Positive Elements.—Theorem. *A continued fraction whose elements are all positive must be either convergent or oscillating.*

For, from Art. **310,** it follows that the odd convergents always increase, and the even convergents always decrease, while any even convergent is greater than any following odd convergent. Hence, in this case the even convergents tend to a limit, and so also with the odd convergents, so that $L\dfrac{A_{2n}}{B_{2n}} = \alpha$ say, and $L\dfrac{A_{2n+1}}{B_{2n+1}} = \beta$ say, where $\alpha \geqq \beta$. If $\alpha = \beta$, the continued fraction is convergent, and if $\alpha > \beta$, the continued fraction oscillates.

316. Theorem.—*If the elements of the continued fraction*

$$b_0 + \frac{1|}{|b_1} + \frac{1|}{|b_2} + \cdots + \frac{1|}{|b_n} + \cdots$$

are all positive, the necessary and sufficient condition for the convergence of the continued fraction is that the series $\sum b_n$ *diverge.*

The necessity of the condition has been shown in Art. **313.** Since every $B_n > 0$, we have $B_n = b_n B_{n-1} + B_{n-2} > B_{n-2}$. The series equivalent to the continued fraction,

$$\frac{A_0}{B_0} + \frac{1}{B_0 B_1} - \frac{1}{B_1 B_2} + \frac{1}{B_2 B_3} - \frac{1}{B_3 B_4} + \cdots + (-1)^{n-1} \frac{1}{B_{n-1} B_n} + \cdots,$$

is an alternating series and the absolute values of its terms are decreasing, so that the alternating series test (Art. **81**) can be applied if we can show that the n^{th} term approaches 0, or

$$\underset{n}{L} B_{n-1} B_n = \infty.$$

From the recursion formulae of Art. **310** we can easily show that

$$B_{2n} > b_1(b_2 + b_4 + b_6 + \cdots + b_{2n}),$$
$$B_{2n+1} > b_1 + b_3 + b_5 + \cdots + b_{2n+1}.$$

Now if $\sum b_n$ diverges, at least one of the increasing sequences B_{2n}, B_{2n+1} must $\to \infty$, so that $\underset{n}{L} B_{n-1} B_n = \infty$, and the equivalent series is convergent, and hence the continued fraction is convergent. Thus the condition is sufficient.[1]

317. Regular Continued Fractions.—A continued fraction is called **regular** (or simple) when all the partial numerators a_i are equal to 1, and all the partial denominators b_i are positive integers, while the first term b_0 may have any integral value; it has therefore the form

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[1] Numerous other tests of convergence for continued fractions have been found, but we shall not be able to treat them here.

$$b_0 + \frac{1|}{|b_1} + \frac{1}{|b_2} + \cdots + \frac{1|}{|b_n} + \cdots$$

It is denoted by $[b_0, b_1, b_2, \ldots, b_n, \ldots]$.

It can be shown that every regular infinite continued fraction converges, and represents an irrational number; conversely, that every irrational number can be developed into a regular continued fraction in one and only one way, and this continued fraction is necessarily infinite.

A regular continued fraction $[b_0, b_1, b_2, \ldots, b_n, \ldots]$ such that $b_{n+k} = b_n$ for every n, that is one of the form

$$[b_0, b_1, \ldots, b_{k-1}, b_0, b_1, \ldots, b_{k-1}, b_0, b_1, \ldots, b_{k-1}, \ldots],$$

in which the elements $b_0, b_1, \ldots, b_{k-1}$ repeat themselves periodically, is called **periodic.**

It can be shown that a periodic regular infinite continued fraction represents a quadratic irrational number, and conversely that the regular continued fraction into which a quadratic irrational can be developed is always periodic.[1]

318. Conditional and Unconditional Convergence.—The question of convergence of a continued fraction is quite different from that of series and products. In series and products it is the ultimate behavior of the terms which determines the convergence or divergence, and the terms at the beginning play no part in determining the convergence, but this is not true in infinite continued fractions.

A convergent continued fraction $b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots$

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[1] For proofs of these results consult PERRON, "Lehre von den Kettenbrüchen," arts. **12, 20.**

is called **unconditionally** convergent, when it remains convergent after an arbitrary number of elements at the beginning have been thrown away, *i.e.*, when the continued fraction

$$b_k + \frac{a_{k+1}|}{|b_{k+1}} + \frac{a_{k+2}|}{|b_{k+2}} + \cdots$$

is convergent for every $k = 0, 1, 2, \ldots$ Otherwise the continued fraction is called **conditionally** convergent.

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CHAPTER XXIV

INFINITE DETERMINANTS

319. Infinite Determinants.—Let $(a_{m,\,n})$ be a double sequence defined for positive integral values of m and n. If we form the determinant of the n^{th} order:

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

and then let $n \to \infty$, we obtain an **infinite determinant**

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \cdots \\ a_{21} & a_{22} & \cdots & a_{2n} & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & \cdots \\ \cdot & \cdot & \cdots & \cdot & \cdots \end{vmatrix}.$$

We may denote it more briefly by $[a_{mn}]_1^\infty$.

The elements a_{ii} (where i takes all values from 0 to ∞) are said to form the **principal diagonal** of the determinant D; the elements a_{ik}, where i is fixed and k takes all values, are said to form the i^{th} **row**; and the elements a_{ik} where k is fixed and i takes all values, the k^{th} **column.** Any element a_{ik} is called a **diagonal or non-diagonal** element, according as $i = k$ or $i \neq k$.

The infinite determinant D is called **convergent, divergent** *(properly or improperly), or* **oscillating** *according as the sequence (D_n) is convergent, divergent, or oscillating.* The determinant is then convergent if $\underset{n}{L} D_n$ exists $= D$, and D is called the **value** of the determinant.

Examples.—(1) The infinite determinant

$$\begin{vmatrix} 1 & \frac{1}{2^2} & \frac{1}{2^4} & \frac{1}{2^8} & \cdots \\ \frac{1}{2} & 1+\frac{1}{2!} & \frac{1}{2^7} & \frac{1}{2^{11}} & \cdots \\ \frac{1}{2^3} & \frac{1}{2^6} & 1+\frac{1}{3!} & \frac{1}{2^{16}} & \cdots \\ \frac{1}{2^5} & \frac{1}{2^{10}} & \frac{1}{2^{15}} & 1+\frac{1}{4!} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

is convergent (by Art. **320**).

(2) The infinite determinant

$$D = \begin{vmatrix} 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ 0 & 0 & 3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

is properly divergent, since $D_n = n!$

(3) The determinant

$$D = \begin{vmatrix} -1 & 0 & 0 & \cdots \\ 0 & -1 & 0 & \cdots \\ 0 & 0 & -1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

is oscillating, since $D_n = (-1)^n$.

The preceding definition of infinite determinant may be generalized by considering a_{mn} defined for all positive and negative integral values of m and n, giving an infinite determinant of the form

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{-1,-1} & a_{-1,0} & a_{-1,1} & a_{-1,2} & \cdots \\ \cdots & a_{0,-1} & a_{0,0} & a_{0,1} & a_{0,2} & \cdots \\ \cdots & a_{1,-1} & a_{1,0} & a_{1,1} & a_{1,2} & \cdots \\ \cdots & a_{2,-1} & a_{2,0} & a_{2,1} & a_{2,2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

320. Theorem.—*An infinite determinant converges if the infinite product formed by the diagonal elements converges absolutely, and the infinite series formed by the non-diagonal elements converges absolutely.*

For convenience we denote the diagonal elements by $1 + a_{ii}$ and the non-diagonal elements by a_{ik} $(i \neq k)$, so that the determinant is

$$\begin{vmatrix} 1 + a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & 1 + a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & 1 + a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}.$$

We are to prove that $L\ D_n$ exists when $\prod\limits_{1}^{\infty}(1 + |a_{ii}|)$ and $\sum\limits_{i,k=1}^{\infty} |a_{ik}|$ both converge.

Since $\sum\limits_{i,k=1}^{\infty} |a_{ik}|$ converges, we see by Art. **224** that $\sum\limits_{k=1}^{\infty} |a_{ik}|$ converges and is $= A_i$ say, and that $\sum\limits_{i=1}^{\infty} A_i$ also converges, hence the product

$$\bar{P} = \prod_{i=1}^{\infty}\left(1 + \sum_{k=1}^{\infty} |a_{ik}|\right)$$

converges (Art. **250**).

Now if we form the finite products

$$P_m = \prod_{i=1}^{m}\left(1 + \sum_{k=1}^{m} a_{ik}\right), \bar{P}_m = \prod_{i=1}^{m}\left(1 + \sum_{k=1}^{m} |a_{ik}|\right),$$

and if in the expansion of P_m, certain terms are replaced by zero and certain other terms have their signs changed, we obtain D_m. Then to each term in the expansion of D_m there corresponds, in the expansion of $\bar{P}_m$ a term of equal or greater absolute value. Now $D_{m+p} - D_m$ represents the sum of those terms of the determinant D_{m+p} which vanish when the elements a_{ik} for $i, k = m + 1, m + 2, \ldots, m + p$, are replaced by zero; and to each of these terms there corresponds a term of equal or greater absolute value in $\overline{P}_{m+p} - \overline{P}_m$. Hence

$$|D_{m+p} - D_m| \leqq \overline{P}_{m+p} - \overline{P}_m.$$

Since $\overline{P}_n \to$ a limit as $n \to \infty$, it follows that

$$\overline{P}_{m+p} - \overline{P}_m < \epsilon$$

for $m \geqq m_0$ and for every p, however small ϵ. Hence

$$|D_{m+p} - D_m| < \epsilon, \; m \geqq m_0,$$

so that by the general principle of convergence, $D_n \to$ a limit, and the determinant D is convergent.

321. Theorem.—*An infinite determinant which converges remains convergent when the elements of any row are replaced by any set of elements which are less in absolute value than some fixed number.*

This may be proved by methods similar to those used in Art. **320.**[1]

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[1] See Whittaker and Watson, "Modern Analysis," art. **2.82.**

CHAPTER XXV

HIGHER TRANSCENDENTS

322. In this chapter we shall give a very brief introduction to the principal higher transcendental functions defined by infinite processes.

GAMMA FUNCTION

323. The Gamma Function.—*The function defined by the infinite product*

$$\Gamma(x) = \frac{e^{-Cx}}{x \cdot \prod_{n=1}^{\infty}\left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}},$$

where C is Euler's constant, is called the **Gamma function.**[1]

We have seen in Art. **261** that the infinite product

$$\prod_{1}^{\infty}\left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}$$

is convergent and defines a function which is continuous for all finite values of x which are not negative integers, and which does not vanish for such values of x.

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[1] It can be proved that when $x > 0$, this function is also represented by the Eulerian integral

$$\int_0^{\infty} e^{-u}u^{x-1}du.$$

(See PIERPONT, "Theory of Functions of Real Variables," vol. II, art. **228.**)

Theorem.—*The Gamma function is a continuous function of x for all finite values of x which are not zero or negative integers, and it vanishes for no finite value of x.*

324. Theorem.—*The Gamma function may be represented by the form*

$$\Gamma(x) = \underset{n}{L} \frac{n^x \cdot (n-1)!.}{x(x+1) \ . \ . \ . \ (x+n-1)}.$$

For if we put

$$G_n(x) = \frac{n^x \cdot (n-1)!}{x(x+1) \ . \ . \ . \ (x+n-1)},$$

we may write

$$G_n(x) = \frac{1}{x} \cdot \frac{e^{x \log n}}{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right) \ . \ . \ . \ \left(1+\frac{x}{n-1}\right)},$$

and since

$$H_n - \log n = C_n \rightarrow C,$$

or

$$\log n = H_n - C_n = 1 + \frac{1}{2} + \frac{1}{3} + \ . \ . \ . \ + \frac{1}{n-1} - C'_n,$$

where

$$C'_n \rightarrow C,$$

we obtain

$$G_n(x) = \frac{1}{x} \cdot \frac{e^{x\left(1+\frac{1}{2}+\frac{1}{3}+ \ . \ . \ . \ +\frac{1}{n-1}\right) - C'_n x}}{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right) \ . \ . \ . \ \left(1+\frac{x}{n-1}\right)}$$

$$= \frac{e^{-C'_n x}}{x} \cdot \prod_{p=1}^{n-1} \frac{e^{\frac{x}{p}}}{1+\frac{x}{p}}.$$

If we now let $n \to \infty$, we get

$$\underset{n}{L} G_n(x) = \frac{e^{-Cx}}{x} \cdot \prod_{1}^{\infty} \frac{e^{\frac{x}{n}}}{1 + \frac{x}{n}} = \Gamma(x).$$

325. Theorem.—*Another expression for the Gamma function is*

$$\Gamma(x) = \frac{1}{x} \cdot \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^x}{1 + \frac{x}{n}}$$

For if we put

$$g_n(x) = \frac{1}{x} \cdot \prod_{p=1}^{n} \frac{\left(1 + \frac{1}{p}\right)^x}{1 + \frac{x}{p}},$$

we find

$$g_n(x) = \frac{1}{x} \cdot \prod_{p=1}^{n} \frac{e^{x \log\left(1 + \frac{1}{p}\right)} e^{-\frac{x}{p}}}{\left(1 + \frac{x}{p}\right) e^{-\frac{x}{p}}}.$$

But

$$\prod_{1}^{\infty} e^{x \log\left(1 + \frac{1}{p}\right) - \frac{x}{p}}$$

and

$$\prod_{1}^{\infty} \left(1 + \frac{x}{p}\right) e^{-\frac{x}{p}}$$

are both convergent. We have then

$$g_n(x) = \frac{\prod\limits_{p=1}^{n} e^{x\left[\log\left(1 + \frac{1}{p}\right) - \frac{1}{p}\right]}}{x \prod\limits_{p=1}^{n} \left(1 + \frac{x}{p}\right) e^{-\frac{x}{p}}}.$$

Now

$$\prod_{p=1}^{n} e^{x\left[\log\left(1+\frac{1}{p}\right)-\frac{1}{p}\right]} = e^{x\sum_{1}^{n}\left[\log\left(1+\frac{1}{p}\right)-\frac{1}{p}\right]} = e^{-C''_n x},$$

since

$$\sum_{p=1}^{n}\left[\log\left(1+\frac{1}{p}\right)-\frac{1}{p}\right] = \log\,(n+1) - H_n$$
$$= -C''_n \rightarrow -C.$$

Then we have

$$g_n(x) = \frac{e^{-C''_n x}}{x\cdot\prod_{1}^{n}\left(1+\frac{x}{p}\right)e^{-\frac{x}{p}}},$$

and letting $n \rightarrow \infty$, we find

$$\underset{n}{L}\, g_n(x) = \frac{e^{-Cx}}{x\cdot\prod_{1}^{\infty}\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}} = \Gamma(x).$$

326. Theorem.—*The Gamma function satisfies the equation*

$$(a) \qquad \Gamma(x+1) = x\cdot\Gamma(x).$$

For if we use the form in Art. **324**, and put

$$\Gamma_n(x) = \frac{n^x(n-1)!}{x(x+1)\;.\;.\;.\;(x+n-1)},$$

we have

$$\Gamma_n(x+1) = \frac{(n-1)!n\cdot n^x}{(x+1)(x+2)\;.\;.\;.\;(x+n)}$$
$$= \frac{nx}{x+n}\cdot\Gamma_n(x),$$

and in the limit when $n \rightarrow \infty$, we get

$$\Gamma(x+1) = x \cdot \Gamma(x).$$

For $x = 1$, we have for every n, $\Gamma_n(1) = 1$, hence $\Gamma(1) = 1$. Applying (a) successively, for m a positive integer, we get

$$\Gamma(m) = 1 \cdot 2 \cdot 3 \;.\;.\;.\; (m-1) = (m-1)!.$$

327. Theorem.—$\Gamma(x) \cdot \Gamma(1-x) = \dfrac{\pi}{\sin \pi x}$.

Using the definition of Art. **323**, we have

$$\Gamma(x)\Gamma(-x) = -\frac{1}{x^2}\prod_{n=1}^{\infty}\left\{\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}\right\}^{-1} \times \prod_{n=1}^{\infty}\left\{\left(1-\frac{x}{n}\right)e^{\frac{x}{n}}\right\}^{-1} = -\frac{1}{x^2}\prod_{n=-\infty}^{+\infty}\left\{\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}\right\}^{-1}.$$

But we have by Art. **265**,

$$\sin \pi x = \pi x \cdot \prod_{n=-\infty}^{+\infty}\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}},$$

and by Art. **326**, $\Gamma(1-x) = -x\Gamma(-x)$, hence we have

$$\Gamma(x) \cdot \Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

This gives a relation between the Gamma function and the circular functions.

WEIERSTRASSIAN ELLIPTIC FUNCTIONS

328. Theorem.—*Let* $2\omega_1$, $2\omega_2$ *be two complex numbers such that the three points* 0, $2\omega_1$, $2\omega_2$ *are not collinear. The double series*

$$(a) \qquad \sum_{m,n}{}' \frac{1}{(2m\omega_1 + 2n\omega_2)^p},$$

where m and n take all positive and negative integral and zero values with the exception of $m = n = 0$ *(indicated by the accent), is convergent for* $p > 2$ *and divergent for* $p \leqq 2$.

Put

$$w_{m,n} = 2m\omega_1 + 2n\omega_2,$$

then as m and n take the values mentioned above, the points $w_{m,n}$ will be the vertices of a network of parallelograms, as in the figure, which completely cover the plane.

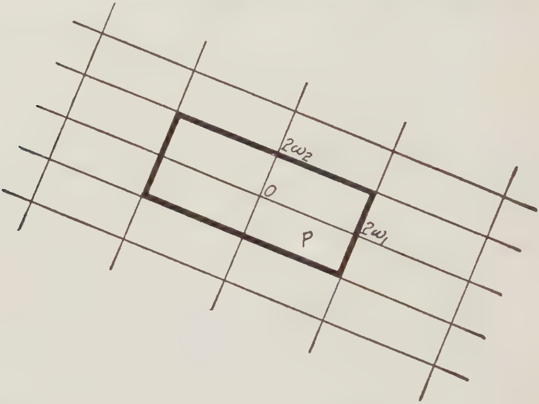

FIG. 9.

Consider the series of absolute values

$$(b) \qquad \sum{}' \frac{1}{|w_{m,n}|^p}.$$

We may replace this double series by the simple series $\sum \beta_n$, where β_1 denotes the sum of the terms of (b) whose indices correspond to points on the first parallelogram P_1 whose center is the origin 0; β_2 is the sum of the terms of (b) whose indices lie on the second parallelogram P_2 about 0, etc.

If d and D are the least and greatest distances of the sides of P_1 from 0, then each of the 8 numbers $w_{m,n}$ which lie on P_1 satisfy the relation.

$$d \leqq |w_{m,n}| \leqq D.$$

Similarly each of the $2 \cdot 8$ numbers $w_{m,n}$ which lie on P_2 satisfy the relation

$$2d \leqq |w_{m,n}| \leqq 2D,$$

etc. Then

$$\frac{8}{D^p} \leqq \beta_1 \leqq \frac{8}{d^p},$$

$$\frac{2 \cdot 8}{(2D)^p} \leqq \beta_2 \leqq \frac{2 \cdot 8}{(2d)^p},$$

etc.

Hence

$$\sum \frac{8s}{(sD)^p} \leqq \sum \beta_s \leqq \sum \frac{8s}{(sd)^p},$$

or

$$\frac{8}{D^p} \sum \frac{1}{s^{p-1}} \leqq \sum \beta_s \leqq \frac{8}{d^p} \sum \frac{1}{s^{p-1}}.$$

Since $\sum \frac{1}{s^{p-1}}$ converges when $p > 2$ and diverges when $p \leqq 2$, our theorem follows at once.

329. The σ Function.—Let u_0 be any arbitrary complex number, and $2\omega_1$ and $2\omega_2$ two complex numbers whose ratio is a pure imaginary. Then the points representing $u_0 + 2m\omega_1 + 2n\omega_2$, where m and n are any positive or negative integers or zero, will be the vertices of a network of parallelograms, all equal to the parallelogram P, which has its vertices at the points u_0, $u_0 + 2\omega_1$, $u_0 + 2\omega_2$, $u_0 + 2\omega_1 + 2\omega_2$. This network will cover the whole plane. The parallelograms such as P are called elementary parallelograms.

Let us put

$$w = 2m\omega_1 + 2n\omega_2,$$

and consider the function defined by the doubly infinite product

$$(a) \qquad \sigma(u) = u \cdot \prod_{m,n}{}'\left(1 - \frac{u}{w}\right)e^{\frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2}},$$

where m and n take all positive and negative integral and zero values except $m = n = 0$.

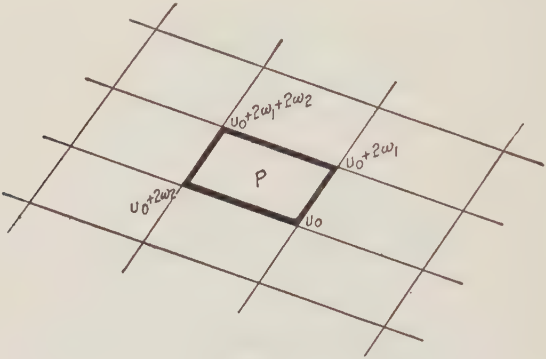

FIG. 10.

This infinite product is convergent. This may be proved by showing that the series of logarithms of the factors is convergent; this series is

$$(b) \qquad \sum{}' \left\{ \log\left(1 - \frac{u}{w}\right) + \frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2} \right\}.$$

If we choose that branch of the logarithm of $\left(1 - \frac{u}{w}\right)$ which tends toward zero when $w \to \infty$, the general term of the series can be written, by developing the logarithm into a series,

$$t_w = -\frac{u^3}{w^3}\left(\frac{1}{3} + \frac{1}{4}\frac{u}{w} + \frac{1}{5}\frac{u^2}{w^2} + \cdot \cdot \cdot\right).$$

The ratio

$$t_w : \left(-\frac{1}{3}\frac{u^3}{w^3}\right)$$

tends toward 1 when $w \to \infty$, so that it suffices to prove the convergence of the series (b) to show that the series

$$\sum{}'\left(-\frac{1}{3}\frac{u^3}{w^3}\right) = -\frac{u^3}{3}\sum{}'\frac{1}{w^3}$$

is convergent, but this has been done in Art. **328.** Hence the infinite product (a) is convergent.

This infinite product (a) being convergent then defines a function $\sigma(u)$, the Weierstrassian sigma function. The function $\sigma(u)$ vanishes only at the points $u = 0$ and $u = w = 2m\omega_1 + 2n\omega_2$; it never becomes infinite for finite values of u. We see that the function has one simple zero in each elementary parallelogram, and only one; all these zeros are similarly placed in the network of parallelograms.

The function $\sigma(u)$ is an odd function, that is

$$\sigma(-u) = -\sigma(u).$$

For since m and n take all values from $-\infty$ to $+\infty$, we may change the signs of m and n in the product (a) without changing its value; we have then

$$\sigma(u) = u \cdot \prod{}'\left(1 + \frac{u}{w}\right)e^{-\frac{u}{w}+\frac{1}{2}\frac{u^2}{w^2}}.$$

If we now change u into $-u$, we get, by comparison with (a), $\sigma(-u) = -\sigma(u)$.

It follows from the infinite product that the ratio $\dfrac{\sigma(u)}{u} \to 1$ when $u \to 0$.

330. The ζ Function.—We define a new function $\zeta(u)$ by taking the logarithmic derivative of $\sigma(u)$:

$$(a) \qquad \zeta(u) = \frac{d}{du}\log\sigma(u) = \frac{\sigma'(u)}{\sigma(u)}$$

$$= \frac{1}{u} + \sum{}'\left\{\frac{1}{u-w} + \frac{1}{w} + \frac{u}{w^2}\right\}.$$

This series shows that the function $\zeta(u)$ has for simple poles[1] all the points $u = 0$, $u = 2m\omega_1 + 2n\omega_2$.

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[1] At a pole a function becomes infinite while its reciprocal becomes zero; at a simple pole the reciprocal has a simple zero.

This ζ function is odd, for $\sigma(u)$ is odd and hence its derivative $\sigma'(u)$ must be even, so that the quotient $\frac{\sigma'(u)}{\sigma(u)}$ is odd: $\zeta(-u) = -\zeta(u)$.

331. The $\wp$ Function.—We now define another new function by taking the negative derivative of the ζ function:

$$(a) \qquad \begin{aligned} \wp(u) &= -\frac{d}{du}\zeta(u) = -\frac{d^2}{du^2}\log\sigma(u) \\ &= \frac{1}{u^2} + \sum{}'\left\{\frac{1}{(u-w)^2} - \frac{1}{w^2}\right\}. \end{aligned}$$

This function being the derivative of an odd function must be even. It has for double poles the points $u = 0$, and $u = 2m\omega_1 + 2n\omega_2$.

The most remarkable property of this function is its double periodicity: that is, it admits the two periods $2\omega_1$, $2\omega_2$, so that

$$\wp(u + 2\omega_1) = \wp(u), \; \wp(u + 2\omega_2) = \wp(u).$$

For, if we form the difference $\wp(u + 2\omega_1) - \wp(u)$, we have

$$\begin{aligned} \wp(u + 2\omega_1) - \wp(u) &= \frac{1}{(u+2\omega_1)^2} - \frac{1}{u^2} \\ &\qquad + \sum{}'\left\{\frac{1}{(u+2\omega_1-w)^2} - \frac{1}{(u-w)^2}\right\} \\ &= \sum\left\{\frac{1}{[u-2(m-1)\omega_1-2n\omega_2]^2} - \frac{1}{[u-2m\omega_1-2n\omega_2]^2}\right\}, \end{aligned}$$

where the last summation is extended over all values of m and n without exception. This last sum is evidently zero, since for proper sets of values of m and n the terms cancel in pairs. We have then

$$(b) \qquad \wp(u + 2\omega_1) = \wp(u),$$

and similarly we find

$$(c) \qquad \wp(u + 2\omega_2) = \wp(u).$$

If we integrate these two relations we find

$$(d) \qquad \zeta(u + 2\omega_1) = \zeta(u) + 2\eta_1,$$

$$(e) \qquad \zeta(u + 2\omega_2) = \zeta(u) + 2\eta_2,$$

where η_1, η_2 denote constants of integration. Making $u = -\omega_1$, and then $u = -\omega_2$, we have

$$\zeta(\omega_1) = \zeta(-\omega_1) + 2\eta_1, \; \zeta(\omega_2) = \zeta(-\omega_2) + 2\eta_2,$$

from which, since $\zeta(u)$ is odd, we obtain

$$(f) \qquad \eta_1 = \zeta(\omega_1), \; \eta_2 = \zeta(\omega_2).$$

If we integrate the relations (d) and (e), we get

$$\log \sigma(u + 2\omega_1) = \log \sigma(u) + 2\eta_1 u + \log c,$$
$$\log \sigma(u + 2\omega_2) = \log \sigma(u) + 2\eta_2 u + \log c',$$

which may be written

$$\sigma(u + 2\omega_1) = ce^{2\eta_1 u}\sigma(u),$$
$$\sigma(u + 2\omega_2) = c'e^{2\eta_2 u}\sigma(u).$$

Making $u = -\omega_1$ and then $u = -\omega_2$, we find

$$c = -e^{2\eta_1\omega_1}, \; c' = -e^{2\eta_2\omega_2}.$$

Hence the function $\sigma(u)$ satisfies the two relations

$$(g) \qquad \begin{cases} \sigma(u + 2\omega_1) = -e^{2\eta_1(u+\omega_1)}\sigma(u), \\ \sigma(u + 2\omega_2) = -e^{2\eta_2(u+\omega_2)}\sigma(u). \end{cases}$$

These functions $\sigma(u)$, $\zeta(u)$, $\wp(u)$, which we have very briefly discussed, form the basis of the Weierstrassian theory of Elliptic Functions.

An elliptic function in general is a doubly-periodic function which is analytic (except at poles), and which has no singularities other than poles in the finite part of the plane. Any elliptic function may be expressed in terms of Weierstrass' functions $\wp$, ζ, σ.

THETA FUNCTIONS

332. The Theta Series.—Let $\tau = t + it'$, $t' > 0$, be a constant, and put $q = e^{\pi i\tau}$, so that $|q| < 1$. Then the series

$$\theta(z) = \sum_{-\infty}^{+\infty} (-1)^n q^{n^2} e^{2niz}$$

may be shown to be convergent by the ratio test, so that it defines a function $\theta(z)$, which may be written $\theta(z, q)$ or $\theta(z|\tau)$ when we wish to indicate the parameters q or τ.

We have

$$\theta(z) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz.$$

We find

$$(a) \qquad \theta(z + \pi) = \theta(z);$$

also

$$\theta(z + \pi\tau) = \sum_{-\infty}^{+\infty} (-1)^n q^{n^2} q^{2n} e^{2niz}$$

$$= -q^{-1}e^{-2zi} \sum_{-\infty}^{+\infty} (-1)^{n+1} q^{(n+1)^2} e^{2(n+1)iz},$$

$$(b) \qquad \therefore \theta(z + \pi\tau) = -q^{-1}e^{-2iz} \cdot \theta(z).$$

From these two results (a) and (b), the function $\theta(z)$ is called a **quasi doubly-periodic function** of z.

It is customary to write $\theta_4(z)$ in place of $\theta(z)$; then three other types of theta functions are defined as follows:

(c) $$\theta_3(z) = \theta_4\left(z + \frac{\pi}{2}\right) = 1 + 2\sum_1^{\infty} q^{n^2} \cos 2nz;$$

(d) $$\left\{\begin{aligned} \theta_1(z) &= -ie^{iz+\frac{1}{4}\pi i\tau}\theta_4\left(z + \frac{\pi\tau}{2}\right) \\ &= -i\sum_{-\infty}^{+\infty}(-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)iz} \\ &= 2\sum_1^{\infty}(-1)^n q^{(n+\frac{1}{2})^2} \sin(2n+1)z; \end{aligned}\right.$$

(e) $$\theta_2(z) = \theta_1\left(z + \frac{\pi}{2}\right) = 2\sum_0^{\infty} q^{(n+\frac{1}{2})^2} \cos(2n+1)z.$$ [1]

The function $\theta_1(z)$ is an odd function, and all the others are even functions.

Written out at length the theta series are:

$$\begin{aligned} \theta_1(z) &= 2q^{\frac{1}{4}} \sin z - 2q^{\frac{9}{4}} \sin 3z + 2q^{\frac{25}{4}} \sin 5z - \ . \ . \ . \ , \\ \theta_2(z) &= 2q^{\frac{1}{4}} \cos z + 2q^{\frac{9}{4}} \cos 3z + 2q^{\frac{25}{4}} \cos 5z + \ . \ . \ . \ , \\ \theta_3(z) &= 1 + 2q \cos 2z + 2q^4 \cos 4z + 2q^9 \cos 6z + \ . \ . \ . \ , \\ \theta_4(z) &= 1 - 2q \cos 2z + 2q^4 \cos 4z - 2q^9 \cos 6z + \ . \ . \ . \ . \end{aligned}$$

Any elliptic function may be expressed in terms of these theta functions.

The theta functions may be expressed as infinite products, as follows:

$$\theta_1(z) = 2q^{\frac{1}{4}}\sin z \cdot \prod_{n=1}^{\infty}(1 - q^{2n}) \cdot \prod_{n=1}^{\infty}(1 - 2q^{2n}\cos 2z + q^{4n}),$$

$$\theta_2(z) = 2q^{\frac{1}{4}}\cos z \cdot \prod_1^{\infty}(1 - q^{2n}) . \prod_1^{\infty}(1 + 2q^{2n}\cos 2z + q^{4n}),$$

$$\theta_3(z) = \prod_1^{\infty}(1 - q^{2n}) \cdot \prod_1^{\infty}(1 + 2q^{2n-1}\cos 2z + q^{4n-2}),$$

$$\theta_4(z) = \prod_1^{\infty}(1 - q^{2n}) \cdot \prod_1^{\infty}(1 - 2q^{2n-1}\cos 2z + q^{4n-2}).$$

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[1] There is considerable divergence of notation for the theta functions.

For proofs of these results, see for instance WHITTAKER and WATSON, "Modern Analysis," pp. 469–470.

HYPERGEOMETRIC FUNCTION

333. The Hypergeometric Series.—The series

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha + 1) \cdot \beta(\beta + 1)}{1 \cdot 2 \cdot \quad \gamma(\gamma + 1)} x^2$$
$$+ \frac{\alpha(\alpha + 1)(\alpha + 2) \cdot \beta(\beta + 1)(\beta + 2)}{1 \cdot 2 \cdot 3 \qquad \gamma(\gamma + 1)(\gamma + 2)} x^3 + \cdot \cdot \cdot$$

is called the **hypergeometric series.** It defines a function $F(\alpha, \beta, \gamma, x)$, called the **hypergeometric function.**

To find the region of convergence of this series, apply the ratio test; we have

$$\left|\frac{a_{n+2}}{a_{n+1}}\right| = \left|\frac{(\alpha + n)(\beta + n)}{(n + 1)(\gamma + n)}\right| \cdot |x|,$$

which $\to |x|$. Thus the series converges absolutely for $|x| < 1$, and diverges for $|x| > 1$.

To test its convergence at the extremities of its interval of convergence, we must make a special investigation. First let $x = +1$. In this case the terms finally become all of one sign, and we have

$$\left|\frac{a_{n+1}}{a_{n+2}}\right| = \frac{n^2 + n(1 + \gamma) + \gamma}{n^2 + n(\alpha + \beta) + \alpha\beta}.$$

Applying Gauss' test (Art. **118**), we find that the series F converges when and only when $\alpha + \beta - \gamma < 0$.

Now let $x = -1$. The terms finally alternate in sign. To show that this alternating series is convergent we must prove that the terms continually decrease and approach zero. We have

$$|a_{n+2}| = \frac{\alpha \cdot \beta}{\gamma} \cdot$$
$$\frac{(\alpha + 1)(\alpha + 2) \cdot \cdot \cdot (\alpha + n)(\beta + 1) \cdot \cdot \cdot (\beta + n)}{(1 + 1)(1 + 2) \cdot \cdot \cdot (1 + n)(\gamma + 1) \cdot \cdot \cdot (\gamma + n)}.$$

Writing

$$\alpha + m = m\left(1 + \frac{\alpha}{m}\right), \beta + m = m\left(1 + \frac{\beta}{m}\right),$$

$$1 + m = m\left(1 + \frac{1}{m}\right), \gamma + m = m\left(1 + \frac{\gamma}{m}\right),$$

we have

$$|a_{n+2}| = \prod_{m=1}^{n} \frac{\alpha\beta}{\gamma} \cdot \frac{\left(1 + \frac{\alpha}{m}\right)\left(1 + \frac{\beta}{m}\right)}{\left(1 + \frac{1}{m}\right)\left(1 + \frac{\gamma}{m}\right)}.$$

But

$$\frac{1}{1 + \frac{1}{m}} = 1 - \frac{1}{m} + \frac{1}{m^2} \cdot \frac{1}{1 + \frac{1}{m}} = 1 - \frac{1}{m} + \frac{\sigma_m}{m^2},$$

where $\sigma_m \to 1$ as $m \to \infty$, and

$$\frac{1}{1 + \frac{\gamma}{m}} = 1 - \frac{\gamma}{m} + \frac{1}{m^2} \cdot \frac{\gamma^2}{1 + \frac{\gamma}{m}} = 1 - \frac{\gamma}{m} + \frac{\tau_m}{m^2},$$

where $\tau_m \to \gamma^2$ as $m \to \infty$. Hence

$$|a_{n+2}| = \frac{\alpha\beta}{\gamma} \cdot \prod_{m=1}^{n} \left(1 + \frac{\alpha}{m}\right)\left(1 + \frac{\beta}{m}\right)\left(1 - \frac{1}{m} + \frac{\sigma_m}{m^2}\right) \times$$

$$\left(1 - \frac{\gamma}{m} + \frac{\tau_m}{m^2}\right) = \frac{\alpha\beta}{\gamma} \cdot \prod_{m=1}^{n} \left(1 + \frac{\alpha + \beta - \gamma - 1}{m} + \frac{\eta_m}{m^2}\right).$$

Then

$$\log |a_{n+2}| = \sum_{m=1}^{n} \log \left(1 + \frac{\alpha + \beta - \gamma - 1}{m} + \frac{\eta_m}{m^2}\right)$$

$$= \sum_{1}^{n} l_m = \lambda_n \text{ say},$$

so that

$$\sum_{1}^{\infty} l_m = \underset{n}{L} \log |a_{n+2}| = \lambda'.$$

Now for a_n to $\to 0$, it is necessary that $\lambda_n \to -\infty$. By the use of Taylor's theorem in finite form, we have

$$\log\left(1 + \frac{\alpha + \beta - \gamma - 1}{m} + \frac{\eta_m}{m^2}\right) = \\ \frac{1}{m}\left[(\alpha + \beta - \gamma - 1) + \frac{\eta_m}{m}\right] \\ - \frac{\theta_m}{m^2}\left[(\alpha + \beta - \gamma - 1) + \frac{\eta_m}{m^2}\right]^2, \qquad 0 < \theta_m < 1.$$

If $\alpha + \beta - \gamma - 1 = 0$, we have

$$l_m = \frac{\eta_m}{m^2}\left(1 - \frac{\theta_m \eta_m}{m^2}\right),$$

which is comparable with the general term of the convergent series $\sum \frac{1}{m^2}$. Thus the series $\sum l_m$ is convergent if $\alpha + \beta - \gamma - 1 = 0$. If $\alpha + \beta - \gamma - 1 \neq 0$, we have $ml_m \to \alpha + \beta - \gamma - 1$, so that $\sum l_m$ is comparable with the divergent series $\sum \frac{1}{m}$. When $\alpha + \beta - \gamma - 1 > 0$, we see that the terms of $\sum l_m$ finally become positive and $\sum l_m$ is properly divergent to $+\infty$. When $\alpha + \beta - \gamma - 1 < 0$, the terms finally become negative and $\sum l_m$ diverges properly to $-\infty$, and $\lambda_n \to -\infty$. Hence, $\lambda_n \to -\infty$ when and only when $\alpha + \beta - \gamma - 1 < 0$, and then $a_{n+2} \to 0$. Now

$$\left|\frac{a_{n+2}}{a_{n+1}}\right| = 1 + \frac{\alpha + \beta - \gamma - 1}{n} + \frac{\delta_n}{n^2},$$

so that when $\alpha + \beta - \gamma - 1 < 0$, we have $|a_{n+2}| < |a_{n+1}|$. Hence, in this case the hypergeometric series is an alternating series satisfying the conditions of Art. **81,** and so is convergent.

We can now sum up in the following:

Theorem.—*The hypergeometric series* $F(\alpha, \beta, \gamma, x)$ *converges absolutely when* $|x| < 1$ *and diverges when* $|x| > 1$. *When* $x = 1$, F *converges only when* $\alpha + \beta - \gamma < 0$, *and then absolutely. When* $x = -1$, *it converges only when* $\alpha + \beta - \gamma - 1 < 0$, *and absolutely when* $\alpha + \beta - \gamma < 0$.[1]

334. The Hypergeometric Function.—The hypergeometric series being a power series with radius of convergence = 1, defines a function $F(\alpha, \beta, \gamma, x)$ when $|x| < 1$.

This function $F(\alpha, \beta, \gamma, x)$ satisfies the differential equation

$$x(1 - x)\frac{d^2u}{dx^2} + [\gamma - (\alpha + \beta + 1)x]\frac{du}{dx} - \alpha\beta u = 0.$$

Many important functions can be expressed in terms of the hypergeometric function; thus

$$\begin{aligned} (1 + x)^n &= F(-n, \beta, \beta, -x), \ (\beta \text{ arbitrary}), \\ \log (1 + x) &= xF(1, 1, 2, -x), \\ e^x &= \underset{\beta}{L} F\left(1, \beta, 1, \frac{x}{\beta}\right), \\ \sin x &= \underset{\alpha, \beta \to \infty}{L} xF\left(\alpha, \beta, \frac{3}{2}, -\frac{x^2}{4\alpha\beta}\right), \\ \cosh x &= \underset{\alpha, \beta \to \infty}{L} F\left(\alpha, \beta, \frac{1}{2}, \frac{x^2}{4\alpha\beta}\right), \end{aligned}$$

etc.

We have also

$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\,\Gamma(\gamma - \beta)},$$

when the real part of $(\gamma - \alpha - \beta)$ is > 0.[2]

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[1] PIERPONT, "Theory of Functions of Real Variables," vol. II.

[2] For proofs of these statements, see WHITTAKER and WATSON, "Modern Analysis," chap. XIV.

LEGENDRE FUNCTIONS

335. Legendre Polynomials.—If $(1 - 2zh + h^2)^{-1/2}$ be expanded in a power series in h, and we write

$$(1 - 2zh + h^2)^{-1/2} = P_0(z) + h.P_1(z) + h^2P_2(z) + h^3P_3(z) + \dots,$$

then we find

$$P_0(z) = 1,\ P_1(z) = z,\ P_2(z) = \tfrac{1}{2}(3z^2 - 1),\ P_3(z) = \tfrac{1}{2}(5z^3 - 3z),$$
$$P_4(z) = \tfrac{1}{8}(35z^4 - 30z^2 + 3),\ P_5(z) = \tfrac{1}{8}(63z^5 - 70z^3 + 15z),\ \dots,$$

and generally

$$P_n(z) = \frac{(2n)!}{2^n \cdot (n!)^2}\left\{z^n - \frac{n(n-1)}{2(2n-1)}z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)}z^{n-4} - \dots\right\}$$
$$= \sum_{r=0}^{m}(-1)^r \cdot \frac{(2n-2r)!}{2^n \cdot r!\,(n-r)!\,(n-2r)!}z^{n-2r},$$

where $m = \tfrac{1}{2}n$ or $\tfrac{1}{2}(n-1)$ according to which is an integer.

These expressions $P_0(z)$, $P_1(z)$, $P_2(z)$, . . . , which are all polynomials in z, are called **Legendre's polynomials.**[1]

It is found that the function $u = P_n(z)$ is a solution of the Legendre differential equation

$$(1 - z^2)\frac{d^2u}{dx^2} - 2z\frac{du}{dz} + n(n+1)u = 0.$$

336. Legendre Functions.—In the preceding paragraph the degree n of the function $P_n(z)$ is a positive integer; $P_n(z)$ can be defined for non-integral values of n.

It is found that the Legendre differential equation is satisfied by the function

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[1] They are also sometimes called Legendre coefficients or zonal harmonics.

$$u = \frac{1}{2\pi i} \int_C \frac{(t^2 - 1)^n}{2^n (t - z)^{n+1}} \, dt,$$

even when n is not a positive integer. This expression is denoted by $P_n(z)$, and is called the **Legendre function of** degree n of the **first kind.**

$P_n(z)$ can be expressed as a hypergeometric function by the formula

$$P_n(z) = F(n + 1, -n, 1, \tfrac{1}{2} - \tfrac{1}{2}z).$$

Legendre's differential equation is also satisfied by the function

$$Q_n(z) = \frac{1}{4i \sin n\pi} \int_D \frac{(t^2 - 1)^n}{2^n (z - t)^{n+1}} dt.$$

This function $Q_n(z)$ is called a **Legendre function** of degree n of the **second kind.**

The importance of the Legendre functions lies in the fact that any arbitrary function can be expanded in a series of Legendre functions. If

$$f(z) = a_0 P_0(z) + a_1 P_1(z) + a_2 P_2(z) + \cdots + a_n P_n(z) + \cdots,$$

we have

$$a_n = (n + \tfrac{1}{2}) \int_{-1}^{+1} f(x) P_n(x) dx.$$

BESSEL FUNCTIONS

337. Bessel Functions.—The series

$$(a)\ J_r(x) = x^r \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{r+2n} \cdot n!(r + n)!} \quad (r = 0, 1, 2, \ldots)$$

converges for every x. For the ratio of two consecutive terms is in absolute value

$$\frac{|x|^2}{2^n (n + 1)(n + r + 1)},$$

which → 0 as $n \to \infty$ for any x.

This power series thus defines a function of x which is everywhere continuous. This function is called a **Bessel function** of order r· sometimes also called a Bessel coefficient.

We have

$$J_0(x) = 1 - \frac{x^2}{2 \cdot 2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots,$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \cdots,$$

$$J_r(x) = \frac{x^r}{2^r \cdot r!}\left\{1 - \frac{x^2}{2^2 \cdot 1 \cdot (r+1)} + \frac{x^4}{2^4 \cdot 1 \cdot 2(r+1)(r+2)} - \cdots\right\}.$$

The function $J_r(x)$ satisfies Bessel's differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{r^2}{x^2}\right)y = 0.$$

By a study of the general solution of this differential equation we are led to a generalization of the preceding Bessel coefficients, giving **Bessel functions** $J_r(x)$ and $Y_r(x)$, defined for non-integral values of r; $J_r(x)$ is called a Bessel function of the **first kind,** and $Y_r(x)$ a Bessel function of the **second kind.**

An arbitrary function $f(x)$ can be expanded in a series of Bessel functions:

$$f(x) = \sum_0^\infty a_n J_n(x).$$

338. Theorem.—*The function* $e^{x \cdot \frac{u - u^{-1}}{2}}$ *can be expanded into a series in terms of Bessel functions of the form*

$$e^{x \cdot \frac{u-u^{-1}}{2}} = \sum_{-\infty}^{+\infty} u^n J_n(x)$$

for every x *and for any* $u \neq 0$.

For

$$e^{x\cdot\frac{u-u^{-1}}{2}} = e^{\frac{xu}{2}} \cdot e^{-\frac{x}{2u}}$$
$$= \left[1 + \frac{xu}{2} + \frac{x^2u^2}{2^2\cdot 2!} + \frac{x^3u^3}{2^3\cdot 3!} + \ . \ . \ . \ \right] \times$$
$$\left[1 - \frac{x}{2u} + \frac{x^2}{2^2\cdot 2!u^2} - \ . \ . \ . \ \right].$$

These series are absolutely convergent for every x and $u \neq 0$, hence they may be multiplied, giving the product

$$e^{x\cdot\frac{u-u^{-1}}{2}} = \left[1 - \left(\frac{x}{2}\right)^2 + \frac{1}{2!2!}\left(\frac{x}{2}\right)^2 - \ . \ . \ . \ \right]$$
$$+ u\left[\frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{3!2!}\left(\frac{x}{2}\right)^5 - \ . \ . \ . \ \right]$$
$$+ u^{-1}\left[\frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{3!2!}\left(\frac{x}{2}\right)^5 - \ . \ . \ . \ \right]$$
$$+ \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .$$
$$= J_0(x) + uJ_1(x) + u^2J_2(x) + \ . \ . \ .$$
$$- \frac{J_1(x)}{u} + \frac{J_2(x)}{u^2} - \frac{J_3(x)}{u^3} + \ . \ . \ .$$

This result shows why $J_n(x)$ is called a Bessel coefficient.

339 Theorem.—$J_{r+1}(x) = \dfrac{2r}{x}\ J_r(x) - J_{r-1}(x), \quad (r > 0).$

For

$$J_{r-1}(x) = \frac{x^{r-1}}{2^{r-1}(r-1)!} + \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+r-1}}{2^{2n+r-1}\cdot n!(n+r-1)!},$$

and

$$J_{r+1}(x) = -\sum_{n=1}^{\infty}(-1)^n \cdot \frac{x^{2n+r-1}}{2^{2n+r-1}(n-1)!(n+r)!},$$

so that

$$J_{r-1}(x) - J_{r+1}(x) = \frac{x^{r-1}}{2^{r-1}(r-1)!} +$$

$$+\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n+r-1}}{2^{2n+r-1}}\left\{\frac{1}{n!\,(n+r-1)!}-\frac{1}{(n-1)!\,(n+r)!}\right\}$$

$$=\frac{x^{r-1}}{2^{r-1}(r-1)!}+r\sum_{1}^{\infty}(-1)^n\cdot\frac{x^{2n+r-1}}{2^{2n+r-1}n!\,(n+r)!}$$

$$=\frac{r}{x}\sum_{0}^{\infty}(-1)^n\frac{x^{2n+r}}{2^{2n+r-1}n!\,(n+r)!}=\frac{2r}{x}J_r(x).$$

340. Theorem.—$2J'_r(x) = J_{r-1}(x) - J_{r+1}(x)$, $(r > 0)$.

For we may differentiate the power series for $J_r(x)$ and get

$$J'_r(x)=\sum_{0}^{\infty}(-1)^n\frac{(2n+r)x^{2n+r-1}}{2^{2n+r}\cdot n!\,(n+r)!}$$

From Art. **339** we find

$$J_{r-1}(x)-J_{r+1}(x)=\frac{x^{r-1}}{2^{r-1}(r-1)!}$$

$$+\sum_{1}^{\infty}(-1)^n\cdot\frac{x^{2n+r-1}}{2^{2n+r-1}}\frac{2n+r}{n!\,(n+r)!}$$

$$=\sum_{0}^{\infty}(-1)^n\frac{(2n+r)x^{2n+r-1}}{2^{2n+r-1}n!\,(n+r)!}=2J'_r(x).$$

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EXERCISES

1. Illustrate the conceptions of upper and lower bounds, limiting points, and upper and lower limits, for the aggregates:

(*a*) $1, 2, 3, \ldots, n, \ldots$

(*b*) $1, 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^{n-1}}, \ldots$

(*c*) $1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$

(*d*) $1, \frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{8}{9}, \ldots$

2. Find the upper and lower bounds, and upper and lower limits, for the sequences:

(*a*) $\frac{1}{m} + \frac{1}{n}$ $(m = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots)$.

(*b*) $2n - 1$ $(n = 1, 2, 3, \ldots)$.

(*c*) $1 + (-1)^n + (-1)^n \cdot \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$.

(*d*) $1 + (-1)^n + \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$.

(*e*) $1 + (-1)^n - \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$.

(*f*) $\dfrac{1}{1 + (-1)^n + \frac{1}{n}}$ $(n = 1, 2, 3, \ldots)$.

3. Find the upper and lower bounds, upper and lower limits, and the limit, if such exists, for the sequences:

(*a*) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots$

(*b*) $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{4}{5}, \ldots$

(*c*) $1, -2, 3, -4, 5, -6, \ldots$

(*d*) $u_{2n} = 1 + \frac{1}{n+1}, u_{2n-1} = n$ $(n = 1, 2, 3, \ldots)$.

(*e*) $u_{4n+1} = \frac{1}{1} + \frac{1}{n+1}, \quad u_{4n+2} = \frac{1}{2} + \frac{1}{n+1}, \quad u_{4n+3} = \frac{1}{3} + \frac{1}{n+1},$
$u_{4n+4} = n$ $(n = 1, 2, 3, \ldots)$.

4. Determine the upper and lower bounds, and upper and lower limits of the following sequences:

(*a*) $u_n = (-1)^n \frac{n}{2n+1}$.

(b) $u_n = (-1)^n \dfrac{n+1}{2n+1}$.

(c) $u_n = n + (-1)^n(2n+1)$.

(d) $u_n = 2n + 1 + (-1)^n \cdot n$.

5. Prove: If $L(u_n \pm v_n) = W$, and $Lu_n = U$, then Lv_n exists and is $= \pm W \mp U$;

If $Lu_nv_n = P$, and $Lu_n = U \neq 0$, then Lv_n exists and is $= \dfrac{P}{U}$;

If $L\dfrac{u_n}{v_n} = Q$, and $Lv_n = V$, then Lu_n exists and is $= V \cdot Q$;

If $L\dfrac{u_n}{v_n} = Q \neq 0$, and $Lu_n = U$, then Lv_n exists and is $= \dfrac{U}{Q}$.

6. Prove: If $Lu_n = U$, then

$$L\frac{u_1v_1 + u_2v_2 + \dots + u_nv_n}{v_1 + v_2 + \dots + v_n} = L\frac{1}{n}(u_1 + u_2 + \dots + u_n).$$

7. Show that Abel's partial summation formula may be written

$$\sum_{n=r}^{s} (a_n - a_{n-1})\, b_n = \sum_{n=r}^{s-1} a_n(b_n - b_{n+1}) - a_{r-1}b_r + a_sb_s.$$

8. Prove: If $Lu_n = U > 0$, then $L\sqrt[n]{u_1u_2 \dots u_n} = U$.

9. Derive Cauchy's theorem (Art. **50**) from Ex. **8**.

10. Prove that if $Ln(a_n - a_{n-1})$ exists, when $La_n = a$, its value must be 0.

11. Find $\underset{x}{L} \log_a \left(1 + \dfrac{1}{x}\right)^x$.

12. Find $L\sqrt[n]{n}$.

13. Find $Ln(\sqrt[n]{a_n} - 1)$, when $La_n = a$. [Use Art. **54**].

14. Find $L\dfrac{1}{n}\sqrt[n]{n!}$. [Use Art. **50**].

15. Find $\underset{x \to 0}{L} \dfrac{(1+x)^m - 1}{x}$. [Use Art. **56**].

16. Find $\underset{n}{L}\dfrac{1^p + 2^p + \dots + n^p}{n^{p+1}}$. [Use Art. **38** and Ex. **5**].

17. Prove that

$$L\left(\frac{1^p + 2^p + \dots + n^p}{n^p} - \frac{n}{p+1}\right) = \frac{1}{2}.$$

18. Show that if $a_p = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dots + \dfrac{1}{p}$, then

$$\frac{a_1 + a_2 + \dots + a_{n-1}}{n} = a_n - 1.$$

19. Show that if $S_p = \dfrac{1}{2 \log 2} + \dfrac{1}{3 \log 3} + \dfrac{1}{4 \log 4} + \dots + \dfrac{1}{p \log p}$, then

$$\underset{n}{L}\left(\frac{S_2 + S_3 + \ . \ . \ . \ + S_{n-1}}{n} - S_n\right) = 0.$$

20. Prove: If l_1x, l_2x, l_3x, . . . denote $\log x$, $\log(\log x)$ or $\log\log x$, $\log\log\log x$, . . . , respectively, then

$$\underset{x}{L}\frac{l_r(x+1)}{l_rx} = 1.$$

[Prove for $r = 1$, then for any r by mathematical induction.]

21. Prove: If l denotes $\log_e(\quad)$, and λ denotes $\log_a(\quad)$, then

$$\underset{x}{L}\frac{\lambda_rx}{l_rx} = \frac{1}{\log_e a}.$$

[Prove by induction.]

22. Prove that

$$\underset{n}{L}\frac{H_n}{l_rn} = \infty \ (r > 1).$$

$$\left[\text{Write } \frac{H_n}{l_2n} = \frac{H_n}{l_1n}\cdot\frac{l_1n}{l_2n}, \ . \ . \ . \ , \frac{H_n}{l_rn} = \frac{H_n}{l_{r-1}n}\cdot\frac{l_{r-1}n}{l_rn}\right].$$

23. Prove that

$$\log z \begin{cases} \leqq z - 1, \\ \geqq 1 - \dfrac{1}{z}, \end{cases} \quad (0 \leqq z \leqq \infty).$$

From this derive

$$\log M_{n+1} - \log M_n \begin{cases} \leqq \dfrac{M_{n+1} - M_n}{M_n}. \\ \geqq \dfrac{M_{n+1} - M_n}{M_{n+1}}. \end{cases}$$

24. Evaluate $\underset{n}{L}e^{-n^a}n^b$, $\underset{n}{L}\,n^{-a}\log n$ $(a > 0,\ b > 0)$.

25. Evaluate the limits: $\underset{x\to0}{L}\left(x^{\frac{1}{\log(e^x-1)}}\right)$, $\underset{x\to1}{L}\ x^{\frac{1}{x-1}} = e$,

$$\underset{x\to1}{L}\ (1 + \log x)^{\frac{1}{x-1}} = e,$$

26. Prove in detail the examples of Arts. **63–64.**

27. Express the results of Arts. **48, 49, 51, 58, 59, 60, 61, 62** in terms of the infinitary notation.

28. Prove that if $f_1(x) = O[g(x)]$ and $f_2(x) = O[g(x)]$, then $f_1(x) + f_2(x) = O[g(x)]$.

29. Prove that if $f(x) = O[ag(x)]$, then $f(x) = O[g(x)]$.

30. Prove that if $f(x) = o[g(x)]$, then $f(x) = O[g(x)]$.

31. Prove that if $f_1(x) \sim f_2(x)$ and $f_2(x) \sim f_3(x)$, then $f_1(x) \sim f_3(x)$.

32. Prove that if $f_1 \succ g_1$, $f_2 \succ g_2$, then $f_1 + f_2 \succ g_1 + g_2$.

33. If $P_m(x)$, $Q_n(x)$ denote polynomials of degrees m, n, whose first coefficients A, B are positive, prove that:

(*a*) $P_m(x) \sim \frac{A}{B} \cdot Q_n(x)$, $(m = n)$,
(*b*) $P_m(x) \succ Q_n(x)$, $(m > n)$,
(*c*) $P_m(x) \sim Ax^m$.

34. Prove: (*a*) $e^x \succ x^k$, $\log x \prec x^k$.
(*b*) $x + a \sin x \sim x$,
(*c*) $x^m = O(e^{kx})$,
(*d*) $\frac{\log x}{x} = O(x^{k-1})$,
(*e*) $H_n \succ 1$, $H_n \sim \log n$,
(*f*) $H_n - \log n \sim C$,
(*g*) $n! \prec n^n$.

35. Write out the first five terms of the following series:

$$\sum_1^\infty \frac{\sqrt{n}}{n+1}, \quad \sum_1^\infty \frac{2n-1}{2^n}, \quad \sum_1^\infty \frac{2n-1}{n^2}, \quad \sum_1^\infty n^2, \quad \sum_1^\infty \frac{1}{2^{n-1}},$$

$$\sum_1^\infty n(n+1), \quad \sum_1^\infty \frac{1}{n!}, \quad \sum_1^\infty (2r)^2, \quad \sum_1^\infty \frac{n}{(2n-1)!}, \quad \sum_1^\infty (-1)^{n+1} \frac{1}{2n-1},$$

$$\sum_1^\infty \frac{x^n}{n!}, \quad \sum_1^\infty \frac{1}{n(n+1)}, \quad \sum_1^\infty (-1)^{n-1} \frac{3^n}{5^{2n-1}},$$

$$\sum_{[illegible]}^\infty \frac{2n-1}{2n} x^{n-2}, \quad \sum_1^\infty (-1)^{k+1} \frac{k}{2^k}$$

36. Find the n^{th} term of each of the following series:

$1 + 2 + 3 + 4 + \ldots$
$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots$
$1^2 + 3^2 + 5^2 + 7^2 + \ldots$
$1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - 4 \cdot 5 + \ldots$
$\frac{3}{5^2} + \frac{3^2}{10^2} + \frac{3^3}{15^2} + \frac{3^4}{20^2} + \ldots$
$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$
$1 - \frac{2!}{2^2} + \frac{3!}{3^2} - \frac{4!}{4^2} + \ldots$
$1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots$
$x + x^3 + x^5 + x^7 + \ldots$

$$\frac{1}{x}+\frac{1}{x^4}+\frac{1}{x^9}+\frac{1}{x^{16}}+\ \cdot\ \cdot\ \cdot$$

$$\frac{x^2}{\sqrt{2}}+\frac{x^4}{\sqrt{4}}+\frac{x^6}{\sqrt{6}}+\frac{x^8}{\sqrt{8}}+\ \cdot\ \cdot\ \cdot$$

$$\frac{1}{2\cdot 3}+\frac{2}{3\cdot 4}+\frac{3}{4\cdot 5}+\frac{4}{5\cdot 6}+\ \cdot\ \cdot\ \cdot$$

$$\frac{1}{2}+\frac{1\cdot 3}{2\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}+\ \cdot\ \cdot\ \cdot$$

37. Prove that $\sum \frac{1}{\sqrt{n}+\sqrt{n-1}}$ is divergent, by writing

$$\frac{1}{\sqrt{n}+\sqrt{n-1}}=\sqrt{n}-\sqrt{n-1},$$

and finding S_n.

38. Test for convergency:

(*a*) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ \cdot\ \cdot\ \cdot$

(*b*) $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ \cdot\ \cdot\ \cdot$

(*c*) $1+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ \cdot\ \cdot\ \cdot$

(*d*) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ \cdot\ \cdot\ \cdot$

(*e*) $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+\ \cdot\ \cdot\ \cdot$

39. Prove that if $M_n \neq 0$ for every n and $LM_n = +\infty$, then $\sum \frac{M_{n+1}-M_n}{M_nM_{n+1}}$ is convergent. [Find S_n.]

40. Test for convergency:

(*a*) $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ \cdot\ \cdot\ \cdot$

(*b*) $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\ \cdot\ \cdot\ \cdot$

(*c*) $\frac{1}{\sqrt{2}}-\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[4]{2}}-\frac{1}{\sqrt[5]{2}}+\ \cdot\ \cdot\ \cdot$

(*d*) $1+\frac{1}{\sqrt{2^3}}+\frac{1}{\sqrt{3^3}}+\frac{1}{\sqrt{4^3}}+\ \cdot\ \cdot\ \cdot$

(*e*) $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+\ \cdot\ \cdot\ \cdot$

41. Compute to four decimal places:

(*a*) $1-\frac{1}{3^2}+\frac{1}{5^3}-\frac{1}{7^4}+\ \cdot\ \cdot\ \cdot$

(*b*) $\frac{1}{2}-\frac{1}{2^3}\cdot\frac{1}{3!}+\frac{1}{2^5}\cdot\frac{1}{5!}-\frac{1}{2^7}\cdot\frac{1}{7!}+\ \cdot\ \cdot\ \cdot$

(*c*) $1-\frac{1}{2^2}+\frac{1}{3^3}-\frac{1}{4^4}+\ \cdot\ \cdot\ \cdot$

(*d*) $\frac{1}{1}\cdot\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1}{3}\cdot\frac{1}{3^3}-\frac{1}{4}\cdot\frac{1}{3^4}+\ \cdot\ \cdot\ \cdot$

42. Test for convergency:

(a) $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{3}{4}} + \sqrt{\frac{4}{5}} + \cdots$

(b) $1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$

(c) $2/1 - 3/2 + 4/3 - 5/4 + 6/5 - \cdots$

(d) $\frac{1}{1 + 2\sqrt{2}} + \frac{1}{1 + 3\sqrt{3}} + \frac{1}{1 + 4\sqrt{4}} + \cdots$

43. Test for convergency:

(a) $\frac{1}{2 + 1} + \frac{1}{2^2 + 1} + \frac{1}{2^3 + 1} + \cdots$

(b) $\frac{1}{a} + \frac{1}{2a + b} + \frac{1}{3a + b} + \cdots$

(c) $\frac{1}{5} + \frac{1}{7} + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 2^2} + \frac{1}{4 \cdot 2^3} + \cdots$

(d) $\frac{1}{a^2} + \frac{1}{a^2 + 1} + \frac{1}{a^2 + 2} + \cdots$

44. Test for convergency:

(a) $\frac{1}{1} + \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + \cdots$

(b) $\frac{1}{1} + \frac{1 \cdot 3}{1 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{1 \cdot 4 \cdot 7} + \cdots$

(c) $\frac{3}{5} + \frac{3 \cdot 5}{5 \cdot 10} + \frac{3 \cdot 5 \cdot 7}{5 \cdot 10 \cdot 15} + \cdots$

(d) $\frac{2}{2 \cdot 3 \cdot 4} + \frac{4}{3 \cdot 4 \cdot 5} + \cdots + \frac{2n}{(n + 1)(n + 2)(n + 3)} + \cdots$

45. Apply Raabe's test to $\sum \frac{1}{n}$.

46. Test the series

$$1 + \frac{3}{2} \cdot \frac{1}{5} + \frac{3 \cdot 5}{2 \cdot 4} \cdot \frac{1}{7} + \cdots + \frac{3 \cdot 5 \cdots 2n + 1}{2 \cdot 4 \cdots 2n} \cdot \frac{1}{2n + 3} + \cdots$$

by applying Gauss' rule.

47. Test for convergency:

(a) $\frac{1}{a(a + b)} + \frac{1}{(a + b)(a + 2b)} + \frac{1}{(a + 2b)(a + 3b)} + \cdots$

(b) $\frac{2}{1 + 2\sqrt{2}} + \frac{3}{1 + 3\sqrt{3}} + \frac{4}{1 + 4\sqrt{4}} + \cdots$

(c) $\frac{1}{2} + \frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{3}{5} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{5} \cdot \frac{1}{2} \cdot \frac{3}{5} + \cdots$

(d) $\frac{2}{2 \cdot 3} + \frac{4}{3 \cdot 4} + \frac{6}{4 \cdot 5} + \cdots$

(e) $\frac{2}{3} + 2\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right)^3 + \cdots$

48. Test for convergency:

(*a*) $\frac{1}{\sqrt{3}}+\frac{1}{\sqrt[3]{3}}+\frac{1}{\sqrt[4]{3}}+\frac{1}{\sqrt[5]{3}}+\cdots$

(*b*) $\frac{2}{4}+\frac{2\cdot 4}{4\cdot 7}+\frac{2\cdot 4\cdot 6}{4\cdot 7\cdot 10}+\cdots$

(*c*) $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\cdots$

(*d*) $\frac{1}{2}+\frac{1}{2^4}+\frac{1}{2^9}+\frac{1}{2^{16}}+\cdots$

(*e*) $\frac{1}{3}+\frac{1}{3}\cdot\frac{2}{5}+\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{3}{7}+\cdots$

49. Test for convergency:

(*a*) $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\cdots$

(*b*) $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\cdots$

(*c*) $\frac{2}{1}-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\cdots$

(*d*) $\frac{2}{3}-\frac{3}{4}\cdot\frac{1}{2}+\frac{4}{5}\cdot\frac{1}{3}-\frac{5}{6}\cdot\frac{1}{4}+\cdots$

(*e*) $1-\frac{2^3}{3!}+\frac{2^5}{5!}-\frac{2^7}{7!}+\cdots$

✗ **50.** Test for convergency:

(*a*) $\sum\frac{\sqrt{n}}{n^2+1}$.

(*b*) $\sum\frac{n^2-(n-1)^2}{n^3+(n+1)^3}$.

(*c*) $\sum\frac{n+1}{n(n+2)}$.

(*d*) $\sum\frac{1}{(2n+1)^n}$.

(*e*) $\sum\frac{n}{(n+1)^n}$.

51. When is $\frac{1}{1+x}+\frac{1}{1+2x^2}+\frac{1}{1+3x^3}+\cdots$ convergent?

52. When is $\frac{1}{1+x}+\frac{1}{1+x^2}+\frac{1}{1+x^3}+\cdots$ convergent?

53. Show that $\frac{a}{1}+\frac{a(a+1)}{2!}+\frac{a(a+1)(a+2)}{3!}+\cdots$ is divergent when $a>0$.

54. Apply Cauchy's integral test to $\sum\frac{1}{n}$.

55. Apply the ratio test and radical test to the series

$$a + b^2 + a^3 + b^4 + \cdot \cdot \cdot \quad (0 < a < b < 1).$$

56. Discuss the convergence of

$$1 - \frac{m}{1} + \frac{m(m-1)}{1 \cdot 2} + \cdot \cdot \cdot + (-1)^n \frac{m(m-1) \cdot \cdot \cdot (m-n+1)}{n!} + \cdot \cdot \cdot$$

57. Discuss the convergence of

$$\sum \frac{1}{n^\alpha (\log n)^\beta}$$

for $\alpha > 1$, $\beta \geqq 0$, and also for $\beta < 0$.

58. Test $\sum \frac{1}{n^k e^{H_n}}$ for convergence for $k = 0$, and for $k \neq 0$, by applying Bertram's tests.

59. Test $\sum q^{n^2}$ and $\sum q^{n^2} x^{2n} (0 < q < 1)$ for convergence.

60. Prove that a necessary condition for convergence of the positive series $\sum a_n$ is

$$\underline{L}a_n = 0, \ \underline{L}na_n = 0, \ \underline{L}na_n l_1 n = 0, \ \underline{L}na_n l_1 n l_2 n = 0, \ \cdot \cdot \cdot$$

61. Test for convergency:

(*a*) $\sum \frac{1+n}{1+n^2}$.

(*b*) $\sum \frac{n^3}{n!}$.

(*c*) $\sum \frac{n^p}{n^p + a}$.

(*d*) $\sum n^{-\frac{n+1}{n}}$.

(*e*) $\sum \frac{1}{n^2 \pm 1}$.

62. Show that $\sum \frac{1}{\left(1 + \frac{1}{n}\right)^{n^2}}$ is convergent.

63. Test $\frac{x}{1+x^2} + \frac{x^2}{1+x^4} + \frac{x^3}{1+x^6} + \cdot \cdot \cdot + \frac{x^n}{1+x^{2n}} + \cdot \cdot \cdot$ for convergency. [Use ratio test.]

64. Test $\sum \frac{1}{(\log n)^n}$ by the radical test.

65. Test $\sum \frac{1}{n(n+1)}$ by Raabe's test.

66. Prove $\frac{1}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6} + \cdot \cdot \cdot$ divergent.

67. Prove that $\sum (-1)^{n-1} \sin \frac{x}{n}$ is conditionally convergent.

68. Test $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{5\cdot 6\cdot 7}+\frac{1}{9\cdot 10\cdot 11}+\frac{1}{13\cdot 14\cdot 15}+\cdots$ by the telescopic method.

69. Test $$\frac{1}{x+1}+\frac{2!}{(x+1)(x+2)}+\frac{3!}{(x+1)(x+2)(x+3)}+\cdots$$ by Raabe's test.

70. Test $1-\frac{1}{1+x}+\frac{1}{2}-\frac{1}{2+x}+\frac{1}{3}-\frac{1}{3+x}+\cdots$ by grouping the terms in pairs and applying Raabe's test.

71. Test $$\frac{x}{x+a_1}+\frac{2!x^2}{(x+a_1)(2x+a_2)}+\frac{3!x^3}{(x+a_1)(2x+a_2)(3x+a_3)}+\cdots$$ where $a_n\to a$, by Raabe's test.

72. Test $\sum n!\left(\frac{x}{n}\right)^n$ by the ratio test.

73. Test for convergency, by the comparison test:

(a) $\sum\frac{1}{(n!)^2}$.

(b) $\sum\frac{1}{3^{n^2}}$.

(c) $\sum\frac{1}{n^n}$.

(d) $\sum\frac{1+2^{n-1}}{3^{n-1}}$.

(e) $\sum\frac{1}{n\cdot n!}$.

(f) $\sum\frac{1}{(2n+1)!}$.

(g) $\sum\frac{n+1}{(n+2)\cdot n!}$.

(h) $\sum\frac{1}{n\cdot 2^{n-1}}$.

(i) $\sum\frac{1}{n\cdot 3^n}$.

(j) $\sum\frac{n}{5n+1}$.

(k) $\sum\frac{n+1}{n}$.

(l) $\sum\frac{2+n}{1+n^2}$.

74. Test for convergency, by the ratio test:

(a) $\sum\frac{3^n}{n!}$.

(b) $\sum\frac{n}{(n+1)!}$.

(c) $\sum\frac{n!}{a^n}$.

(d) $\sum\frac{n}{2^n}$.

(e) $\sum\frac{(n-1)!}{n^2}$.

(f) $\sum\frac{2^n}{n}$.

75. If for the series $\sum u_n$ and $\sum v_n$ we have $u_n-u_{n+1}\leq v_n-v_{n+1}$, for every n, can we conclude that $\sum u_n$ converges if $\sum v_n$ converges? Ex. $u_n=1+\frac{1}{n^3}$, $v_n=\frac{1}{n^2}$.

76. Test $$\left(1+\frac{1}{1}\right)^2-\left(1+\frac{1}{2}\right)^2+\left(1+\frac{1}{3}\right)^2-\cdots+(-1)^{n-1}\left(1+\frac{1}{n}\right)^2+\cdots$$ for convergence.

77. Find the radius of convergence:

(a) $x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}+\cdots$

(*b*) $1 + \frac{x}{2} + \frac{x^2}{2 \cdot 2^2} + \frac{x^3}{3 \cdot 2^3} + \ldots$

(*c*) $1 + 2x + 3x^2 + 4x^3 + \ldots$

(*d*) $1 - \frac{x^2}{\sqrt{2}} + \frac{x^4}{\sqrt{4}} - \frac{x^6}{\sqrt{6}} + \ldots$

(*e*) $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \frac{x^4}{17} + \ldots$

78. Find the radius of convergence:

(*a*) $\sum (-1)^{n+1}(3x)^n.$

(*b*) $\sum \frac{n^2x^n}{2^n}.$

(*c*) $\sum \frac{n^px^n}{n!}.$

(*d*) $\sum \frac{x^{n+1}}{3^n(n+2)}.$

(*e*) $\sum nx^n.$

79. For what values of x does

$$x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \ldots + \frac{1 \cdot 3 \cdot 5 \ldots 2n-1}{2 \cdot 4 \cdot 6 \ldots 2n} \cdot \frac{x^{2n+1}}{2n+1} + \ldots$$

converge? For $x = 1$, test by Gauss' rule.

80. Find the radius of convergence:

(*a*) $1 + \frac{x}{1^2} + \frac{x^2}{2^3} + \frac{x^3}{3^4} + \ldots$

(*b*) $1 + x + 2!x^2 + 3!x^3 + \ldots$

(*c*) $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$

(*d*) $1 + \frac{3}{5}x + \frac{3 \cdot 6}{5 \cdot 8}x^2 + \frac{3 \cdot 6 \cdot 9}{5 \cdot 8 \cdot 11}x^3 + \ldots$

(*e*) $1 + mx + \frac{m^2x^2}{2!} + \frac{m^3x^3}{3!} + \ldots$

81. Find the radius of convergence:

(*a*) $$1 + \frac{3}{5}x + \frac{3 \cdot 5}{5 \cdot 10}x^2 + \ldots + \frac{3 \cdot 5 \ldots (2n+1)}{5 \cdot 10 \ldots 5n}x^n + \ldots$$

(*b*) $2(2x)^2 + 3(2x)^3 + 2(2x)^4 + 3(2x)^5 + \ldots$

(*c*) $mx + \frac{m(m-2)}{2!}x^2 + \frac{m(m-2)(m-4)}{4!}x^3 + \ldots$

(*d*) $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \frac{x^4}{4 \cdot 5} + \ldots$

(*e*) $1 + 3x + 5x^2 + 7x^3 + 9x^4 + \ . \ . \ .$

82. Test for convergency:

(*a*) $\dfrac{3x}{x+4} + \dfrac{1}{2}\left(\dfrac{3x}{x+4}\right)^2 + \dfrac{1}{3}\left(\dfrac{3x}{x+4}\right)^3 + \ . \ . \ .$

(*b*) $\dfrac{x}{x^2+1} + \left(\dfrac{x}{x^2+1}\right)^2 + \left(\dfrac{x}{x^2+1}\right)^3 + \ . \ . \ .$

(*c*) $\dfrac{x-1}{x} + \dfrac{1}{2}\left(\dfrac{x-1}{x}\right)^2 + \dfrac{1}{3}\left(\dfrac{x-1}{x}\right)^3 + \ . \ . \ .$

(*d*) $\dfrac{9}{2} + \dfrac{10}{3}x + \dfrac{25}{12}x^2 + \ . \ . \ . + \dfrac{2n^2+5n+2}{n(n+1)}x^{n-1} + \ . \ . \ .$

83. Test for convergence

$$1 + \frac{1}{2}x + \frac{1}{2\cdot 3}x^2 + \frac{1}{2^2\cdot 3}x^3 + \frac{1}{2^2\cdot 3^2}x^4 + \ . \ . \ . + \frac{1}{2^n 3^{n-1}}x^{2n-1} + \frac{1}{2^n\cdot 3^n}x^{2n} + \ . \ . \ .$$

The ratio test does not apply; but compare with $\sum\left(\dfrac{1}{2}\right)^n$. Also find radius of convergence by radical test.

84. Test $\sum\limits_{-\infty}^{+\infty} e^{nx+pn^2}$ by applying the radical test. Prove it absolutely convergent for $p < 0$ and divergent for $p \geqq 0$.

85. Show that $\sum\limits_{1}^{\infty}\dfrac{(-1)^{n-1}}{n^\alpha}$, $\sum\limits_{1}^{\infty}\dfrac{(-1)^{n-1}}{n^\beta}$, $\alpha > 0$, $\beta > 0$, $\alpha + \beta > 1$, can be multiplied by Cauchy's rule.

86. Test the series $\dfrac{1}{x} + \sum\limits_{-\infty}^{+\infty}{}'\left(\dfrac{1}{x+n} - \dfrac{1}{n}\right)$ for absolute convergence, by the comparison test, using $\sum \dfrac{1}{n^2}$.

87. Test for convergency:

(*a*) $\sum \sin n\alpha$.

(*b*) $\sum \dfrac{1}{n}\sin\dfrac{\alpha}{n}$.

(*c*) $\sum \sin\dfrac{\alpha}{n}$. [Find Lnu_n.]

88. Test the series

$$\sin z + \frac{1}{2}\sin^2 z + \frac{1}{3}\sin^3 z + \ . \ . \ . + \frac{1}{n}\sin^n z + \ . \ . \ .$$

89. Prove $\sin 2x = 2 \sin x \cos x$ by multiplying the series for $\sin x$, $\cos x$.

90. By multiplication of series, prove that

(*a*) $(1 + x + x^2 + \ . \ . \ . \ )^2 = (1 + 2x + 3x^2 + 4x^3 + \ . \ . \ . \ ) = (1 - x)^{-2}$.

(*b*) $2 \sin^2 x = 1 - \cos 2x$.

91. From the expansions for $\sin x$ and $\cos x$, derive

$$\csc x = \frac{1}{x} + \frac{x}{3!} + \frac{7x^3}{3 \cdot 5!} + \ . \ . \ . \ ,$$

$$\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \ . \ . \ .$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ . \ . \ .$$

92. Show that if $\sum |u_n(x)|$ is uniformly convergent in an interval (a, b), so also is $\sum u_n(x)$.

93. Prove that the series

$$(1 - x) + \left(x - \frac{x^2}{2}\right) + \left(\frac{x^2}{2} - \frac{x^3}{3}\right) + \ . \ . \ .$$

is uniformly convergent in $-1 \leq x \leq 1$.

94. Discuss the convergence of the series

$$(x - xe^{-x^2}) + (xe^{-x^2} - 2xe^{-2x^2}) + \ . \ . \ . \ ,$$

and prove that it is not uniformly convergent in any interval extending up to the origin.

95. Discuss the uniform convergence of the series

$(1 - x) - x(1 - x) + x^2(1 - x) + \ . \ . \ . \ + (-1)^n x^n(1 - x) + \ . \ . \ .$

96. Show that $\sum \dfrac{e^{nx} - 1}{2^n e^{nx}}$ is uniformly convergent in $(0, \infty)$ by using Weierstrass' test.

97. Show that $\sum\limits_0^\infty (a_n \cos nx + b_n \sin nx)$ is uniformly convergent in $(-\infty, +\infty)$ provided $\sum a_n$, $\sum b_n$ are absolutely convergent. [Apply Weierstrass' test.]

98. Test the series $1 + 2x + 3x^2 + 4x^3 + \ . \ . \ .$ for uniform convergence.

99. Show that

$$1 + \frac{1}{2}k^2 \sin^2 \phi + \frac{1 \cdot 3}{2 \cdot 4}k^4 \sin^4 \phi + \ . \ . \ . \ = \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}},$$

where $0 < k < 1$, is uniformly convergent for all values of ϕ.

100. Show that

$$K \equiv \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \ . \ . \ . \ \right].$$

101. Test the series

$$1 + \frac{\sin x}{1^{5/2}} + \frac{\sin^2 x}{2^{5/2}} + \frac{\sin^3 x}{3^{5/2}} + \cdots$$

for uniform convergence.

102. Show that the series for which $S_n(z) = nze^{-nz^2}$ cannot be integrated term by term.

103. Differentiate the series

$$\sin z - \tfrac{1}{2}\sin 2z + \tfrac{1}{3}\sin 3z - \tfrac{1}{4}\sin 4z + \cdots$$

term by term. The given series represents the function $\frac{z}{2}$ for $-\pi < z < \pi$; but the series of derivatives does not represent the derivative of $\frac{z}{2}$, for it does not even converge for values of z other than $z = 0$. Prove.

104. Show that

$$\frac{1}{1\cdot 3\cdot 5} + \frac{z}{3\cdot 5\cdot 7} + \frac{z^2}{5\cdot 7\cdot 9} + \cdots + \frac{z^{n-1}}{(2n-1)(2n+1)(2n+3)} + \cdots$$

converges uniformly in a circle within the unit circle. Integrate term by term.

105. Show that $z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$ converges uniformly in any finite region. Differentiate and justify.

106. Discuss the uniform convergence of

$$1 + \frac{z}{\lambda} + \frac{z^2}{\lambda^2} + \frac{z^3}{\lambda^3} + \cdots + \frac{z^n}{\lambda^n} + \cdots,$$

$$1 + \frac{z}{\lambda} + \frac{z^4}{\lambda^2} + \frac{z^9}{\lambda^3} + \cdots + \frac{z^{n^2}}{\lambda^n} + \cdots.$$

107. Show that the two series

$$\frac{1}{2}\cdot\frac{1-z}{1} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{(1-z)^2}{2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{(1-z)^3}{3} + \cdots,$$

$$\frac{1}{2}\cdot\frac{z-1}{1} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{(z-1)^2}{2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{(z-1)^3}{3} + \cdots,$$

have the same region of convergence.

108. Test for uniform convergence:

$$1 + x\sin\theta + x^2\sin 2\theta + x^3\sin 3\theta + \cdots, \quad |x| < X < 1,$$

$$1 + \frac{\sin x}{1^{5/2}} + \frac{\sin^2 x}{2^{5/2}} + \frac{\sin^3 x}{3^{5/2}} + \cdots, \qquad |x| \leq X < \infty.$$

109. Show that the series

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \frac{z^4}{1-z^8} + \cdots$$

converges for all values of z except for values on the unit circle.

110. Prove that

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots$$

is convergent for all values of x, but not uniformly convergent in an interval including the origin.

111. Determine the region of convergence of

$$\frac{z-1}{z} + \frac{1}{2}\left(\frac{z-1}{z}\right)^2 + \frac{1}{3}\left(\frac{z-1}{z}\right)^3 + \cdots$$

112. Derive the expansion of

$$\int \frac{z\,dz}{\cos z} = \frac{z^2}{2} + \frac{z^4}{4\cdot 2!} + \frac{5z^6}{6\cdot 4!} + \cdots$$

113. Discuss the convergence of the Laurent series

$$-\frac{1}{z^n} - \cdots - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \cdots$$
$$- \frac{1}{2^{n+1}} z^n - \cdots$$

114. Carry the reversion of the following series to terms in the fifth power:

(*a*) $y = \sin x = x - \frac{1}{6}x^3 + \cdots$
(*b*) $y = e^x = 1 + x + \frac{1}{2}x^2 + \cdots$
(*c*) $y = 2x + 3x^2 + 4x^3 + \cdots$

115. Examine the double sequence $u_{mn} = \dfrac{mn}{m^3 + n^3}$ for double and repeated limits.

116. Examine the double sequence $u_{mn} = (-1)^m \cdot \dfrac{m^2 n^3}{m^3 + n^6}$ for double and repeated limits.

117. Examine the double sequences $a_{mn} = \dfrac{mn}{m^2+n^2}$, $a_{mn} = \dfrac{m^2n^2}{m^3+n^3}$ for limits and convergence.

118. In the double series

$$\begin{array}{l} 1 + 2 + 4 + 8 + \cdots \\ -\frac{1}{2} - 1 - 2 - 4 - \cdots \\ -\frac{1}{4} - \frac{1}{2} - 1 - 2 - \cdots \\ -\frac{1}{8} - \frac{1}{4} - \frac{1}{2} - 1 - \cdots \\ - \cdots\cdots\cdots\cdots\cdots, \end{array}$$

examine the row- and column-series and the diagonal-series and the repeated summations for convergence. Is the double series itself convergent?

119. As in Ex. **118**, examine the double series

$$\begin{array}{l} 0 + 1 + 0 + 0 + 0 + \cdots \\ -1 + 0 + 1 + 0 + 0 + \cdots \end{array}$$

$$\begin{aligned}
&+0-1+0+1+0+\ .\ .\ .\\
&+0+0-1+0+1+\ .\ .\ .\\
&+0+0+0-1+0+\ .\ .\ .\\
&+\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .
\end{aligned}$$

120. Examine the double series

$$\begin{aligned}
&\ \ 2+0-1+0+0+0+\ .\ .\ .\\
&+0+2+0-1+0+0+\ .\ .\ .\\
&-1+0+2+0-1+0+\ .\ .\ .\\
&+0-1+0+2+0-1+\ .\ .\ .\\
&+0+0-1+0+2+0+\ .\ .\ .\\
&+0+0+0-1+0+2+\ .\ .\ .\\
&+\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .
\end{aligned}$$

121. Show that

$$\sum_0^\infty \sum_0^\infty x^m y^n = \frac{1}{(1-x)(1-y)},\ |x| < 1,\ |y| < 1.$$

122. Discuss the convergence of the following double series:

$$(a)\ \sum_{m,n} (-1)^{n-1}\frac{r^m}{n} \qquad (b)\ \sum_{m,n} (-1)^{n-1}\frac{r^m}{n!}$$

$$(c)\ \sum_{m,n} \frac{x^m y^n}{m+n} \qquad (d)\ \sum_{m,n} \frac{1}{(m+n)^2}, \qquad (e)\ \sum_{m,n} \frac{1}{m^2-n^2}.$$

123. Investigate the double series with the partial sums

$$S_{m,n} = \left(\frac{m+1}{m+2}\right)^{n+1},$$

and determine the terms u_{mn}.

124. Test the following infinite products for convergence:

$$(a)\ \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\ .\ .\ .$$

$$(b)\ \left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{4^2}\right)\ .\ .\ .$$

$$(c)\ (1+x)(1+x^2)(1+x^4)(1+x^8)\ .\ .\ .$$

$$(d)\ \left(1-\frac{1}{\log 2}\right)\left(1-\frac{1}{(\log 4)^2}\right)\left(1-\frac{1}{(\log 8)^3}\right)\ .\ .\ .$$

$$(e)\ \left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\ .\ .\ .$$

$$(f)\ \left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{3}}\right)\left(1-\frac{1}{\sqrt{4}}\right)\ .\ .\ .$$

$$(g)\ \left(1+\frac{x}{1}\right)\left(1-\frac{x^2}{2}\right)\left(1+\frac{x^3}{3}\right)\left(1-\frac{x^4}{4}\right)\ .\ .\ .$$

125. Show that $\prod_1^\infty\left(1+\frac{1}{n^2}\right)$ is absolutely convergent, $\prod_2^\infty\left(1-\frac{1}{n}\right)$ is zero divergent, $\prod\limits_{1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)$ is divergent, and $\prod\left(1-\frac{x^{n-1}}{n}\right)$ is absolutely convergent if $x < 1$ and has zero for its value if $x = 1$.

126. Examine $\prod\left(1-\frac{x^n}{n}\right)$, $\prod\left(1-\frac{x}{n^2}\right)$.

127. Prove that

$$\prod_{2}^{\infty}\left(1-\frac{2}{n(n+1)}\right)=\frac{1}{3}.$$

128. Show that

$$\prod_{2}^{\infty}\frac{n^3-1}{n^3+1}=\frac{2}{3}$$

129. Investigate the convergence of

$$\prod_{2}^{\infty}\left(1+\frac{(-1)^n}{\log n}\right).$$

130. Prove that: If for every x in a region B, we have $|u_n(x)| < A_n$, and if $\sum A_n$ is convergent, then $\prod\ (1+u_n(x))$ converges uniformly in B.

131. Prove that $\prod\limits_{2}^{\infty}\left(1+\frac{(-1)^n}{n^x}\right)$ is conditionally convergent for $1 \geqq x = \frac{1}{2}$.

132. Prove that

$$\prod\frac{1}{1-\frac{1}{p^x}}=\sum_{1}^{\infty}\frac{1}{n^x},$$

where p takes all prime values 2, 3, 5, 7, . . .

133. Prove that

$$\prod_{1}^{\infty}\left(1-\frac{x}{a^n}\right)=1+\sum_{1}^{\infty}\frac{(-1)^n x^n}{(a-1)(a^2-1)\ .\ .\ .\ (a^n-1)},\quad |a|>1$$

134. Evaluate the infinite integrals:

$$\int_{-\infty}^{+\infty}\frac{8a^3dx}{x^2+4a^2}=4\pi a^2,\qquad \int_{0}^{2a}\frac{x^{3/2}dx}{\sqrt{2a-x}}=\frac{3\pi a^2}{2}.$$

135. Test for convergence:

$$(a)\ \int_{a}^{\infty}\frac{dx}{x\sqrt{1+x^2}},\qquad (b)\ \int_{a}^{\infty}\frac{x^2dx}{(a^2+x^2)^2},\qquad (c)\ \int_{0}^{\infty}\frac{dx}{1-x^4},$$

(*d*) $\int_0^1 \frac{\log x dx}{1 - x}$.

136. Prove that $\int_0^\infty e^{-ax} \cos bx dx$ is absolutely convergent when $a > 0$.

137. Prove that $\int_0^\infty \frac{\cos x}{x} dx$ is divergent.

138. Develop $\frac{1}{2}x$ into a Fourier series valid from 0 to π.

139. Find the Fourier series representing $x + x^2$ in the interval $-\pi < x < \pi$.

140. Show that if x is not a multiple of 2π, the series

$$1 + \cos x + \cos 2x + \cos 3x + \cos 4x + \cdots$$

is summable $(C, 1)$ with Sum $\frac{1}{2}$.

141. Show that the infinite continued fraction

$$\frac{1}{1} \dot{+} \frac{1}{3} \dot{+} \frac{4}{5} \dot{+} \frac{9}{7} \dot{+} \cdots \dot{+} \frac{(n-1)^2}{2n-1} \dot{+} \cdots$$

is equivalent to the harmonic series $\sum \frac{1}{n}$ and is therefore divergent.

142. Show that the infinite continued fraction

$$\frac{1}{1} \dot{+} \frac{1}{0} \dot{+} \frac{1}{1} \dot{+} \frac{4}{0} \dot{+} \frac{4}{1} \dot{+} \frac{9}{0} \dot{+} \frac{9}{1} \dot{+} \cdots \dot{+} \frac{n^2}{0} \dot{+} \frac{n^2}{1} \dot{+} \cdots$$

is equivalent to the series

$$\frac{1}{1} - \frac{1}{1} + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n} + \cdots$$

143. Show that the continued fraction $\left[\frac{n^p}{1}\right]_{n=1}^{\infty}$ converges if $0 \leq p \leq 2$; that $\left[\frac{x^n}{1}\right]_{n=1}^{\infty}$ converges if $0 < x \leq 1$.

144. Transform the series

$$\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + \cdots + \frac{(-1)^n}{x+n} + \cdots$$

into an infinite continued fraction.

145. Expand e^x, $\sin x$, $\cos x$ into an infinite continued fraction.

146. Prove that for p a positive integer,

$$\underset{x \to -p}{L} (x + p)\Gamma(x) = \frac{(-1)^p}{p!}.$$

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